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SIMPLE CONNECTIVITY OF p-GROUP COMPLEXES*

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To John Thompson on the occasion of his receipt of the Wolf Prize

ABSTRACT

We investigate the simple connectivity of p-subgroup complexes of finite groups.

Let G be a finite group and p a prime. The **commuting graph** $\Lambda_p(G)$ for G at p is the graph on the set of subgroups of G of order p whose edges are the pairs of commuting subgroups, and the **commuting complex** for G at p is the clique complex $K_p(G) = K(\Lambda_p(G))$ of the commuting graph; that is the simplicial complex whose simplices are the cliques of $\Lambda_p(G)$. The commuting complex has the same homotopy type as the Brown complex and the Quillen complex for G at p. The latter complexes have received a fair amount of attention; see for example [14], [18], and [10].

In this paper we begin a systematic study of the question: For which finite groups G and prime divisors p of the order of G is the commuting complex $K_p(G)$ simply connected? Modulo a conjecture on the simple connectivity of certain minimal complexes, we reduce the problem of deciding simple connectivity to the corresponding problem for simple groups. This latter problem can presumably be solved. Moreover we establish our Conjecture in almost all cases.

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CONJECTURE: *Let G be a finite group such that $G = AF^*(G)$, where A is an elementary abelian p-subgroup of rank at least 3 and $F^*(G)$ is the direct product of the A-conjugates of a simple component L of G of order prime to p. Then $K_p(G)$ is simply connected.*

Given a graph Λ and a vertex x of Λ, write $\Lambda(x)$ for the set of vertices distinct from x and adjacent to x in Λ.

THEOREM 1: *Assume the Conjecture and let G be a finite group, p a prime divisor of the order of G, and $\Lambda = \Lambda_p(G)$. Assume $\Lambda(x)$ is connected for all $x \in \Lambda$ and let $\bar{G} = G/O_{p'}(G)$. Then exactly one of the following holds:*

(1) *$K_p(G)$ is simply connected.*
(2) *$\bar{G} = \bar{G}_1 \times \bar{G}_2$ and $\bar{G}_i$ has a strongly p-embedded subgroup for $i = 1$ and 2.*
(3) *$\bar{G} = \bar{X}(\bar{G}_1 \times \bar{G}_2)$, for some $X \in \Lambda$, $p = 3, 5$, $\bar{G}_1 \cong L_2(8)$, $Sz(32)$, respectively, $\bar{G}_2$ is a nonabelian simple group with a strongly p-embedded subgroup, and X induces outer automorphisms on $\bar{G}_i$ for $i = 1$ and 2.*
(4) *$\bar{G}$ is almost simple and $K_p(\bar{G})$ and $K_p(F^*(\bar{G}))$ are not simply connected.*

THEOREM 2: *Let G be a finite group, p a prime divisor of the order of G, and assume $O_p(G) = 1$, $\Lambda = \Lambda_p(G)$ is connected, and $H_1(K_p(G)) = 0$. Then $m_p(G) > 2$ and $\Lambda(x)$ is connected for each $x \in \Lambda_p(G)$.*

THEOREM 3: *Assume G and L satisfy the hypotheses of the Conjecture and that the Conjecture holds in proper sections of G. Then*

(1) *If L is of Lie type and Lie rank at least 2 then $K_p(G)$ is simply connected.*
(2) *If $L \cong L_2(q)$ with q even then $K_p(G)$ is simply connected.*
(3) *If L is an alternating group then $K_p(G)$ is simply connected.*
(4) *If L is a Mathieu group then $K_p(G)$ is simply connected.*

Theorems 1 and 2 say that, modulo the Conjecture and a short list of exceptions, $K_p(G)$ is simply connected if and only if $m_p(G) > 2$ and $\Lambda(x)$ is connected for each $x \in \Lambda_p(G)$. Moreover if $m_p(G) > 2$ then $\Lambda(x)$ is connected for all $x \in \Lambda_p(G)$ unless G, p is one of the exceptions listed in sections 7 and 8.

The following observations expand upon these points:

Remarks:

(1) If $O_p(G) \neq 1$ then G is contractible and hence simply connected (cf. Lemma 2.2 in [14]). Thus the restriction that $O_p(G) = 1$ in Theorem 2 causes no loss of generality.

(2) It is well known that $\Lambda_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup (cf. 44.6 in [1]). Moreover we know all groups with strongly p-embedded subgroups (cf. 6.2). Thus the restriction in Theorem 2 that Λ be connected results in no loss of generality, and the groups in Cases (2) and (3) of Theorem 1 are completely described.

(3) Recall a simplicial complex is simply connected if and only if its fundamental group is trivial, while the first homology group of the complex is the abelianization of its fundamental group. Thus the hypothesis in Theorem 2 that $H_1(K_p(G)) = 0$ is weaker than simple connectivity. So Theorem 2 says that the hypothesis in Theorem 1 that $\Lambda(x)$ be connected for each $x \in \Lambda$ is necessary for simple connectivity, and that if $O_p(G) = 1$ and $K_p(G)$ is simply connected then $m_p(G) > 2$.

(4) The condition $\Lambda(x)$ connected has various equivalent formulations; see for example 6.3. Further sections 7 and 8 describe all finite groups G with $m_p(G) \geq 3$ such that $\Lambda(x)$ is disconnected for some $x \in \Lambda$. Thus Theorems 1 and 2 do indeed constitute a fairly complete reduction to the simple case, modulo the Conjecture.

(5) Recall that from the Classification of the finite simple groups, each nonabelian simple group L is an alternating group, a group of Lie type, or one of the 26 sporadic groups. Thus Theorem 3 reduces a verification of the Conjecture to the case where L is of Lie type and Lie rank 1 (i.e. $L \cong L_2(q)$, $U_3(q)$, $Sz(q)$, or ${}^2G_2(q)$) or L is one of the 21 sporadic groups which are not Mathieu groups. Further to handle one of the remaining sporadic groups L using 11.5, it suffices to exhibit a family $\mathcal{F}$ of subgroups such that the geometric complex $\mathcal{C}(G, \mathcal{F})$ defined by $\mathcal{F}$ is a simply connected, residually connected flag complex. For example this is done for the Lyons group in [7].

(6) For some simple groups G and primes p we determine when $K_p(G)$ is simply connected. For example 7.3, 7.6, 7.7, 8.5, and 8.6 describe those simple groups and primes for which $K_p(G)$ is *not* simply connected because $\Lambda(x)$ is disconnected for some vertex x. On the otherhand if G is of Lie type in characteristic p then by 5.5, $K_p(G)$ is simply connected if and only if G is of Lie rank at least 3.

(7) If G is sporadic then for most primes p, $m_p(G) \leq 2$, so $K_p(G)$ is not simply connected. In the remaining cases it is likely that usually $K_p(G)$ has the

same homotopy type as the flag complex of the p-local geometry of G (*cf.* [4]) and that the p-local geometry is Cohen-Macaulay in the sense of [14]. For example this is proved in [7] for the Lyons group when $p = 3$.

The proof of our Theorems depends upon the theory developed in [6] and [8] for studying the simple connectivity of simplicial complexes. In particular the reader is referred to these references for notation and terminology. The proof of Theorem 2 uses results in [19] and was suggested by Yoav Segev; it is an improvement on my original proof.

1. Preliminary Lemmas

(1.1): *Let p be a prime, G an almost simple finite group, $F^*(G) = L$, and $|G : L| = p$. Then one of the following holds:*

(1) *$L = X(q^p)$ is of Lie type and $G = LX$ where $X \cong \mathbf{Z}_p$ induces field automorphisms on L.*

(2) *$L = L_n^\epsilon(q)$ with $q - \epsilon \equiv n \equiv 0 \mod p$ and G induces inner-diagonal-field automorphisms on L. In particular $m_p(L) \geq \min\{2, n-2\}$.*

(3) *$p = 3$, $L \cong E_6^\epsilon(q)$, $q \equiv \epsilon \mod 3$, and G induces inner-diagonal-field automorphisms on L. In particular $m_3(L) \geq 5$.*

(4) *$p = 3$, $L \cong D_4(q)$ or ${}^3D_4(q)$ and $m_3(G) \geq 2$.*

(5) *$p = 2$ and $m_2(G) \geq 2$.*

Proof: These are well known facts about the automorphism groups of the simple groups. See for example 7.4 in [12]. ∎

(1.2): *Let G be a finite group, $L \leq G$ with $|G : L| \leq 2$, L quasisimple with $Z(L) \neq 1 = O(G)$, $P \in Syl_2(G)$, and $SCN_3(P) = \varnothing$. Then*

(1) *$L \cong SL_2(q)$ or $Sp_4(q)$, q odd, $SL_4^\epsilon(q)$, $q \equiv -\epsilon \mod 4$, or $A_n/\mathbf{Z}_2$, $7 \leq n \leq 11$.*

(2) *If $m_2(C_P(t)) \leq 2$ for some involution $t \in P \cap L$ then $L \cong A_7/\mathbf{Z}_2$, $SL_2(q)$, or $Sp_4(q)$.*

Proof: As $SCN_3(P) = \varnothing$, P has sectional 2-rank at most 4 (cf. [11]). Then as $O_2(L) \neq 1$, the discussion in Section 2 of Part III of [11] shows $L \cong SL_2(q)$ or $Sp_4(q)$, q odd, $SL_4^\epsilon(q)$, $q \equiv -\epsilon \mod 4$, or $A_n/\mathbf{Z}_2$, $7 \leq n \leq 11$. Extensions of $Sz(8)$ and M_{12} are eliminated as $SCN_3(P) = \varnothing$; *cf.* Lemma 5.1 on page 148 of [11] for M_{12}.

So (1) is established. To prove (2) we may take $G = L$ and $t \in P$ an involution with $m_2(C_P(t)) \leq 2$.

Suppose $L \cong A_n/\mathbf{Z}_2$, for $8 \leq n \leq 11$. As $G = L$ and $SCN_3(P) = \varnothing$, $n \neq 8$ or 9; *eg.* otherwise L has a subgroup $L_3(2)/E_{16}$ containing P. Similarly if $n = 10$ or 11 L has a subgroup H with $P \leq H$ and $|H : K| = 2$ with $K \cong A_8/\mathbf{Z}_2$, so $t \in H - K$ and we check that for each such involution, $m_2(C_P(t)) > 2$.

Finally let $L \cong SL_4^\epsilon(q)$. Then $P = E(P_1 \times P_2)$ with P_i quaternion and $E = \langle a, b\rangle \cong E_4$, with $\langle a\rangle P_i$ semidihedral and $P_1^b = P_2$. Hence each involution $t \in P$ is P-conjugate to an element of P_1P_2 or E and thus $m_2(C_P(t)) > 2$.

Recall the **order complex** of a poset X is the simplicial complex with vertex set X and simplices the finite chains in X. We write $\mathcal{O}(X)$ for the order complex of X, although often we abuse notation and simply write X for this complex.

(1.3): *(Bouc) Let X be a finite poset, and for $B \subseteq X$ and $x \in X$, let $B(\geq x) = \{b \in B : b \geq x\}$, $B(> x) = \{b \in B : b > x\}$, $\mathcal{O}(X)$ the order complex of X, and $f_X(B)$ the set of all $x \in X$ such that $\mathcal{O}(B(\geq x))$ is contractible. Then*

(1) *$B \subseteq f_X(B)$ and if $f_X(B) = X$ and $B \subseteq Y \subseteq X$ then $\mathcal{O}(B)$, $\mathcal{O}(Y)$, and $\mathcal{O}(X)$ have the same homotopy type.*

(2) *Let X^* consist of those $x \in X$ such that $\mathcal{O}(X(> x))$ is not contractible. Then $X = f_X(X^*)$, so $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ have the same homotopy type for each subset Y of X containing X^*.*

Proof: This is Proposition 2 and 4 in Bouc, [10]; as the proof is omitted in [10], we give one here.

First for $b \in B$, b is the least element of $B(\geq b)$, so $b \in f_X(B)$. Also if $B \subseteq Y \subseteq X$ and $X = f_X(B)$, then $Y = f_Y(B)$, so to prove (1), by transitivity of homotopy equivalence it suffices to prove B and X have the same homotopy type. But if $\iota : B \to X$ is the inclusion map then $\iota^{-1}(X(\geq x)) = B(\geq x)$, so Proposition 1.6 in [14] completes the proof. See also 4.3 in [8] for a proof of Proposition 1.6.

Thus (1) is established and it remains to prove (2). We induct on the order of X. If X has order 1 then $X^* = X$, so as $X^* \subseteq f_X(X^*)$, the lemma holds in this case.

Let $x \in X$ and $Z = X(\geq x)$. Then $Z(\geq z) = X(\geq z)$ for each $z \in Z$. But if $Z \neq X$ then by induction Z and $Z^* = X^*(\geq x)$ have the same homotopy type. Thus as Z is contractible, so is $X^*(\geq x)$, so $x \in f_X(X^*)$.

Hence we may assume X has a least element x_0 and it remains to show $x_0 \in f_X(X^*)$; that is we must show X^* is contractible. Now if $x_0 \in X^*$ then X^* has a least element and hence is contractible. On the otherhand if $x_0 \notin X^*$ then $Y = X - \{x_0\} = X(> x_0)$ is contractible and $X^* = Y^*$, so by induction, X^* has the same homotopy type as Y, and hence is contractible.

The **height** of an element x in a finite poset P is $h(x) = dim(P(\leq x))$.

(1.4): *Let $f : P \to Q$ be a map of posets such that*

(1) *$f^{-1}(Q(\leq a))$ is $min\{1, h(a)-1\}$-connected for each $a \in Q$.*

(2) *For $a \in Q$, $Q(> a)$ is connected if $h(a) = 0$ while if $h(a) = 1$ then either $Q(> a) \neq \varnothing$ or $f^{-1}(Q(\leq a))$ is simply connected.*

Assume Q is simply connected. Then P is simply connected.

Proof: It is an easy exercise to show P is connected. To show P is simply connected we use an argument of Quillen in Theorem 9.1 of [14]. Recall the notion of a **local system** on a simplicial complex or poset in section 2 of [8] and [14].

For $a \in Q$ let $\theta(a) = f^{-1}(Q(\leq a))$. Let F be a local system on P and for $a \in Q$ define $E(a) = \lim_{x \in \theta(a)} F(x)$ if $h(a) \neq 0$, while if $h(a) = 0$ set $E(a) = F(x_a)$ for some choice of $x_a \in \theta(a)$. For $x \in \theta(a)$ let $E_{x,a} : F(x) \to E(a)$ be the natural map $u \mapsto \bar{u}$ (*cf.* 1.8 in [8]) if $h(a) \neq 0$. On the otherhand if $h(a) = 0$ let $E_{x,a} = F_p$ for some path p from x to x_a in $\theta(b)$ and some $b > a$. Such paths exist as $\theta(b)$ is connected by (1). Further if $b < b'$ and p' is a path for b' then $F_p = F_{p'}$ by 1.11 in [8] as $\theta(b')$ is simply connected by (1). Thus as $Q(> a)$ is connected by (2), $E_{x,a}$ is independent of the choice of p and b.

Next if $a < b$ define $E_{a,b} : E(a) \to E(b)$ to be the natural map on limits when $h(a) \neq 0$, while if $h(a) = 0$ let $E_{a,b} = E_{x_a,b}$. Observe

(3) If $a < b$ and $x \in \theta(a)$ then $E_{x,b} = E_{a,b} \circ E_{x,a}$.

(4) If $x < y \in \theta(a)$ then $E_{x,a} = E_{y,a} \circ F_{x,y}$.

Now F_p is an isomorphism by 1.4 [8], so if $h(a) = 0$ then $E_{x,a}$ is an isomorphism. If $h(a) \geq 2$ or $h(a) = 1$ and $Q(> a) = \varnothing$, then by (1) and (2), $\theta(a)$ is simply connected, so $E_{x,a}$ is an isomorphism by 1.8.2 in [8]. Finally if $h(a) = 1$ and $Q(> a) \neq \varnothing$ there is $b > a$. Notice $E_{x,a}$ is a surjection as $\theta(a)$ is connected. So as $E_{x,b}$ is a bijection, (3) says $E_{x,a}$ is a bijection.

So in any case $E_{x,a}$ is a bijection. Then if $a < b$, $E_{a,b} = E_{x,b} \circ E_{x,a}^{-1}$ is a bijection by (3). Further if $a < b < c$ then $E_{b,c} \circ E_{a,b} = (E_{x,c} \circ E_{x,b}^{-1}) \circ (E_{x,b} \circ E_{x,a}^{-1}) =$

$E_{x,c} \circ E_{x,a}^{-1} = E_{a,c}$ for $x \in \theta(a)$. So E is a local system on Q.

Finally we claim that if $p = x_0 \cdots x_r$ is a path in P then $E_{x_r,f(x_r)} \circ F_p = E_{f(p)} \circ E_{x_0,f(x_o)}$. For $r = 0$ this is trivial and for $r = 1$ it follows from (3) and (4). Finally for $r \geq 1$ it follows from the case $r = 1$ by an easy induction on r.

In particular if P is not simply connected then by 1.9 in [8] there is a local system F for P and a cycle p in P with $F_p \neq \mathrm{id}$. But then $E_{f(p)} = F_p^{E_{x_0,f(x_0)}} \neq \mathrm{id}$, so Q is not simply connected by 1.11 in [8], a contradiction.

2. The join of complexes

Let D and L be simplicial complexes. Recall the **join** of D and L is the simplicial complex $D \vee L$ whose vertex set is the disjoint union of the vertex sets of D and L and whose k-simplices are the disjoint union $s \vee t$ of an i-simplex s of D and a j-simplex t of L with $-1 \leq i, j$ and $k = i + j + 1$, subject to the convention that $\varnothing$ is the unique (-1)-dimensional simplex of D and L.

(2.1):

(1) *$D \vee L$ is connected if and only if D and L are nonempty or D is connected or L is connected.*

(2) *If D and L are nonempty then $D \vee L$ is simply connected if and only if D or L is connected.*

Proof: Let $K = D \vee L$ and regard D and L as subcomplexes of K. Observe each vertex of D is adjacent to each vertex of L, so (1) is trivial. Assume therefore that D and L are nonempty.

We recall from 1.11 in [8] that a simplicial complex K is simply connected if and only if each cycle p in the graph of K is in the closure of the 2-simplices of K. In that event we say p is **trivial** and write $p \sim 1$. We appeal to various results in [6] and [7] to implement this observation, and use the notation and terminology from those references.

In particular if neither D nor L is connected then K has squares but not triangles, so K is not simply connected. Thus we may assume L is connected.

Suppose $p = x_0 \cdots x_n$ is a cycle in L and let $d \in D$. Then $\{d, x_i, x_{i+1}\}$ is a 2-simplex of K and p is in the closure of such simplices, so p is trivial. Similarly each cycle in D is trivial.

Next $d(a, x) = 1$ for all $a \in D$ and $x \in L$, so the diameter of the graph of K is 2. Thus by 3.3 in [6], it suffices to show each r-gon $p = x_0 \cdots x_r$ is trivial for

$r \leq 5$. By the previous paragraph we may assume p is not contained in D or L. Suppose $p = xyzx$ is a triangle. Then we may take $x, y \in L$ and $z \in D$. But then $\{x, y, z\}$ is a simplex of K and hence $xyzx \sim 1$.

If $p = xyzwx$ is a square then $d(x, z) = 2 = d(y, w)$ as p is not contained in D or L, so we may take $x, y \in D$ and $w, y \in L$. Hence as L is connected, 1.4 in [7] says $p \sim 1$.

Finally if p is a pentagon then we may take $x_0 \in D$ and as $d(x_0, x_i) = 2$ for $i = 2, 3$, these vertices are also in D. Hence as p is not contained in D, we may assume $x_1 \in L$. Then as $x_3 \in D$, $d(x_1, x_3) = 1$, contradicting p a pentagon.

Recall if C, C' are chain complexes then the **tensor product** $C \otimes C'$ is the chain complex with

$$(C \otimes C')_m = \bigoplus_{i+j=m} C_i \otimes C'_j$$

with $\partial_m = \sum_{i+j=m} \partial_i \otimes 1 + (-1)^i (1 \otimes \partial_j)$. Similarly the **torsion product** $C * C'$ is the chain complex with

$$(C * C')_m = \bigoplus_{i+j=m} Tor(C_i, C_j)$$

with $\partial_m = \partial_i * 1 + (-1)^i (1 * \partial_j)$. Recall the **Kunneth formula**; *cf.* p.228 in Spanier [16]:

(2.2): *If C and C' are nonnegative free chain complexes then for each m,*

$$H_m(C \otimes C') \cong (H(C) \otimes H(C'))_m \oplus (H(C) * H(C'))_{m-1}.$$

Let $C(D)$ be the chain complex of D with coeficients in $\mathbf{Z}$ and the usual boundry map ∂. Define $\hat{C}(D)$ to be the chain complex with $\hat{C}_{i+1}(D) = C_i(D)$ and $\hat{\partial}_{i+2} = \partial_{i+1}$ for each $i \geq 0$, while $\hat{C}_0(D) = \mathbf{Z}x_0$ and $\hat{\partial}_1 : x \mapsto x_0$ for all vertices x of D. Observe:

(2.3): $\tilde{H}_i(D) = H_{i+1}(\hat{C}(D))$ *for each i.*

(2.4): $\hat{C}(D \vee L) \cong \hat{C}(D) \otimes \hat{C}(L)$.

Proof: Let $V = \hat{C}(D) \otimes \hat{C}(L)$ and $U = \hat{C}(D \vee L)$. Then

$$V_m = \bigoplus_{(s,t) \in \Delta(m)} \mathbf{Z}(s \otimes t)$$

where $\Delta(m)$ is the set of pairs (s,t) with s an i-simplex of D, t a j-simplex of L, and $i+j+2=m$, subject to the convention that $\varnothing$ is the unique (-1)-simplex of D, L, respectively. Similarly

$$U_m = \bigoplus_{(s,t)\in\Delta(m)} \mathbf{Z}(s \vee t)$$

subject to the convention $s \vee \varnothing = s$ and $\varnothing \vee t = t$.

Moreover the boundry map for V is described above, while the map for U is $s \vee t \mapsto \partial s \vee t + (-1)^i (s \vee \partial t)$. Hence the map $s \otimes t \mapsto s \vee t$ defines an isomorphism of chain complexes.

(2.5): *If D and L are nonempty then*

$$\tilde{H}_n(D \vee L) \cong (\tilde{H}(D) \otimes \tilde{H}(L))_{n-1} \oplus (\tilde{H}(D) * \tilde{H}(L))_{n-2}$$

for all $n \geq 0$.

Proof: This follows from 2.2, 2.3, and 2.4.

(2.6): *If $n, m \geq -1$, D is n-connected, and L is m-connected, then $D \vee L$ is $n+m+2$-connected and $\tilde{H}_{n+m+3}(D \vee L) \cong \tilde{H}_{n+1}(D) \otimes \tilde{H}_{m+1}(L)$.*

Proof: As D is n-connected and L is m-connected, $\tilde{H}_i(D) \cong \tilde{H}_j(L) \cong 0$ for $i \leq n$ and $j \leq m$. Thus $(\tilde{H}(D) \otimes \tilde{H}(L))_k \cong (\tilde{H}(D) * \tilde{H}(L))_k \cong 0$ for $k < n+m+2$. Then apply 2.5 to see that the homology is as claimed. Further $n+m+2 \geq 1$ if and only if $n \geq 0$ or $m \geq 0$. So if $n+m+2 \geq 1$ then $D \vee L$ is simply connected by 2.1.

3. Geometric complexes

Define a **geometric complex** over a finite index set I to be a simplicial complex K whose graph Γ is a geometry over I (In the sense of Tits [17]; see also section 3 of [1].) and such that each simplex of K is contained in a chamber of Γ which is a simplex of K. We also denote the type function of Γ by $\tau : \Gamma \to I$.

Recall that a geometric complex K is **residually connected** if the residue of each of its simplices of corank at least 2 is connected. The **residue** of a simplex s is just the link $Link_K(s)$ (*cf.* Section 3 of [8]) of s at K regarded as a geometric complex over $I - \tau(s)$.

Example: Let G be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of G. Then we have the geometric complex $\mathcal{C}(G, \mathcal{F})$ over I whose set of objects of type i is the coset space G/G_i and with $\{G_j x_j : j\}$ a simplex if and only if $\bigcap G_j x_j \neq \varnothing$. See sections 3 and 41 in [1] for a discussion of this example. ∎

Example: Let Γ be a geometry over I. The **flag complex** of Γ is the clique complex of Γ regarded as a graph. Notice the flag complex is a geometric complex over I if and only if each flag of Γ is contained in a chamber. ∎

Write $Rad(K)$ for the subcomplex of all simplices s of K such that $K = st_K(s)$. For $J \subseteq I$ the **truncation** of K at J is the subcomplex of all simplices s with $\tau(s) \subseteq J$, regarded as a geometric complex over J.

Define the product $\Gamma \bowtie \Delta$ of geometries Γ and Δ over I to be the geometry over I with $(\Gamma \bowtie \Delta)_i = \Gamma_i \times \Delta_i$ and with $(x,a) * (y,b)$ if and only if $x * y$ and $a * b$. Define the **geometric product** $K \bowtie K'$ of geometric complexes K and K' over I to be the geometric complex with geometry $\Gamma \bowtie \Gamma'$ and chambers

$$C \bowtie C' = \{(a, a') : a \in C,\ a' \in C' \text{ and } \tau(a) = \tau(a')\},$$

where C, C' are chambers of K, K', respectively. We usually write xa for the vertex $(x,a) \in K \bowtie K'$.

(3.1): *Let D, L be geometric complexes over I. Then $D \bowtie L$ is connected if and only if D and L are connected.*

Proof: Let $K = D \bowtie L$. The projection $\phi_D : K \to D$ is a surjective morphism of graphs, so if K is connected, so is D.

Conversely assume D and L are connected and let ax, by be vertices of K. Then there exists a path $p = a_0 \cdots a_n$ from a to b in D. Let C, C' be chambers in D, L containing b, x, respectively. Let $P = u_0 \cdots u_n$ be the path in K with $\phi_D(P) = p$ and $\phi_L(P) \subseteq C'$. Then P is a path from ax to bz for some $z \in C'$ of type $\tau(b)$. Let $q = x_0 \cdots x_m$ be a path from z to y in L and $Q = v_0 \cdots v_m$ the path in K with $\phi_L(Q) = q$ and $\phi_D(Q) \subseteq C$. Then PQ is a path from ax to by.

(3.2): *Let D, L be geometric complexes over I, $K = D \bowtie L$, and $\phi = \phi_D$ and $\psi = \phi_L$ the projection maps of K on D and L, respectively. Then*

(1) *If p is a cycle in D, x is a vertex in K with $\phi(x) = org(p)$, and C is a chamber in L with $\psi(x) \in C$ then there exist a cycle q in K with origin x, $\phi(q) = p$, and $\psi(q) \subseteq C$.*

(2) *The map* $\alpha : \pi_1(K,x) \to \pi_1(D,\phi(x)) \times \pi_1(L,\psi(x))$ *defined by* $\alpha(\tilde{r}) = (\phi_*(\tilde{r}), \psi_*(\tilde{r}))$ *is a surjective group homomorphism.*

Proof: Part (1) is just an argument from the proof of the previous lemma. In (2), $\tilde{r}$ denotes the equivalence class of the cycle r under the relation $\sim$ defined by the closure of the 2-simplices of K, and $\tilde{p}$ is defined similarly for each cycle p of D. Then $\phi_*(\tilde{r}) = \tilde{\phi}(r)$. By 1.2 in [7], ϕ_* is well defined, and then of course α is a group homomorphism. Further if p, q are cycles in D, L with origin $\phi(x)$, $\psi(x)$, we claim there exists a cycle r of K with origin x and $\phi(r) \sim p$ and $\psi(r) \sim q$. Once we prove this claim we have our surjectivity.

First if C, C' are chambers of L, D containing $\psi(x), \phi(x)$, then by (1) there are cycles s, t in K with $\phi(s) = p$, $\psi(s) \subseteq C$ and $\psi(t) = q$, $\phi(t) \subseteq C'$. Let $r = s \cdot t$. Then $\phi(r) = \phi(s)\phi(t) = p \cdot \phi(t)$ and as $\phi(t) \subseteq C'$ and C' is contractible, $\phi(t) \sim 1$, so $\phi(r) \sim p$. Similarly $\psi(r) \sim q$.

(3.3):

(1) *If* $D \bowtie L$ *is simply connected then* D *and* L *are simply connected.*

(2) $D \bowtie L$ *is residually connected if and only if* D *and* L *are residually connected.*

(3) $D \bowtie L$ *is a flag complex if and only if* D *and* L *are flag complexes.*

(4) *If* D *and* L *are simply connected, residually connected flag complexes of dimension at least 2, then* $D \bowtie L$ *is simply connected.*

Proof: Let $K = D \bowtie L$ and $\phi = \phi_D$. Observe that for simplices s of D and t of L of the same type, the residue $Link_K(st) = Link_D(s) \bowtie Link_L(t)$, so by 3.1, K is residually connected if and only if D and L are residually connected. Thus (2) holds and similarly (3) holds.

Suppose K is simply connected. Then by 3.1, D is connected. Further by 3.2, $\phi_* : \pi_1(K) \to \pi_1(D)$ is a surjection. But as K is simply connected, $\pi_1(K) = 0$, so $\pi_1(D) = 0$ and hence D is simply connected.

Conversely assume D and L are simply connected and residually connected of dimension at least 2. We prove K is simply connected by induction on $m = |I - \tau(Rad(D))|$. If $m = 0$ then D is a chamber so ϕ_L is an isomorphism and hence K is simply connected. Thus we may take $m \geq 1$.

Let $I = \{1, \ldots, n\}$ with $|D_i| = 1$ for $i > m$. Pick C to be a chamber of L and for $i > m$ let $z_i = \tau^{-1}(i) \cap C$ and $a_i = \tau^{-1}(i) \cap D$. Then $S = \{a_i z_i : i > m\}$ is a flag of K of type $J = \{m+1, \ldots, n\}$.

For $d \in D$ define $\theta(d) = \phi^{-1}(st_D(d))$ if $\tau(d) \notin J$ and $\theta(d) = S$ if $\tau(d) \in J$. Claim $\theta(d)$ is simply connected. This is clear if $\tau(d) \in J$ as S is contractible. If $\tau(d) \notin J$ then $\theta(d) = st_D(d) \bowtie L$ and $st_D(d)$ is contractible and hence simply connected. Further $\{d, a_i : i > m\} \subseteq Rad(st_D(d))$, so $\theta(d)$ is simply connected by induction on m.

Next $\theta(d) \cap \theta(e)$ is connected for all 1-simplices $\{d,e\}$ of D. For if $\tau(d) \notin J$ then $\theta(d) \cap \theta(e) = S$, while if $\tau(d), \tau(e) \notin J$ then $\theta(d) \cap \theta(e) = st(\{d,e\}) \bowtie L$ is connected by 3.1. Here we use the fact that D is a flag complex to conclude $sd(d) \cap sd(e) = sd(\{d,e\})$.

Finally $\theta(d) \cap \theta(e) \cap \theta(f) \neq \varnothing$ for all 2-simplices $\{d,e,f\}$ as $S \subseteq \theta(d) \cap \theta(e) \cap \theta(f)$ and if $S = \varnothing$ then $m = n$ and $\{dx, ey, fz\} \subseteq \theta(d) \cap \theta(e) \cap \theta(f)$ for $x, y, z \in C$ of type $\tau(d), \tau(e), \tau(f)$.

Therefore θ is a 1-approximation of K by D in the sense of [8]. For if $s = \{ax, by\}$ is a 1-simplex of K then $s \subseteq \theta(a)$ unless $\tau(a) \in J$, while if $\tau(a), \tau(b) \in J$ then as $J \neq I$, $s \subseteq \theta(v)$ for $\tau(v) \notin J$.

Let a, b be vertices of D and $cx \in \theta(a) \cap \theta(b)$. Then either

(i) $\tau(a)$ or $\tau(b)$ is in J and $cx \in S$, or

(ii) $a, b \in c^{\perp}$ and $\tau(a), \tau(b) \notin J$.

In either case $a, b \in c^{\perp}$.

Let $\mathcal{F}(cx) = \{d \in D : cx \in \theta(d)\}$. To complete the proof of the lemma using Theorem 3 of [8], we verify that $\mathcal{F}$ satisfies the hypotheses of that Theorem. If $cx \in S$ then $cx \in \theta(d)$ for all $d \in D$ so $\mathcal{F}(cx) = D$ is connected and in particular a and b are in the same connected component of $\mathcal{F}(cx)$.

Thus we may assume (ii) holds. Then $\mathcal{F}(cx) = st_D(c) - Rad(D)$. In particular if $\tau(c) \notin J$ then acb is a path in $\mathcal{F}(cx)$. Thus we may take $\tau(c) \in J$. Then $\mathcal{F}(cx) = D - Rad(D)$. But as D is residually connected, $D - Rad(D)$ is connected when $m > 1$. Therefore we may asume $m = 1$. In particular $\tau(a) = \tau(b)$.

Let L' be the truncation of L at J. As $dim(L) \geq 2$, L is residually connected, and as $m = 1$, L' is connected. But $\theta(a) \cap \theta(b) = Rad(D) \bowtie L' \cong L'$, and hence is connected. Thus cx is connected to $cz_i \in S$ in $\theta(a) \cap \theta(b)$ and a is connected to b in $\mathcal{F}(cz_i) = D$. So the proof is complete.

(3.4): *If D and L are residually connected flag complexes then the map α of 3.2.2 is an isomorphism $\pi_1(D \bowtie L, x) \cong \pi_1(D, \phi(x)) \times \pi_1(L, \psi(x))$.*

Proof: Let $\delta : \hat{D} \to D$ and $\lambda : \hat{L} \to L$ be the universal coverings of D, L (*cf.*

section 1 of [8]) and let $\hat{K} = \hat{D} \bowtie \hat{L}$. Define $\xi : \hat{K} \to K$ by $\xi(uv) = \delta(u)\lambda(v)$. Then ξ is a simplicial map and as δ and λ are surjective, so is ξ. Similarly $\xi_{uv} : st_{\hat{K}}(uv) \to st_K(\xi(uv))$ is an isomorphism as $st_{\hat{K}}(uv) = st_{\hat{D}}(u) \bowtie st_{\hat{L}}(v)$ and $st_K(\xi(uv)) = st_D(\delta(u)) \bowtie st_L(\lambda(v))$. So ξ is a connected covering of K. On the otherhand by 3.3, $\hat{K}$ is simply connected, so ξ is even the universal covering.

Recall the discussion of local systems in section 1 of [8]. Notice $\xi^{-1}(xy) = \delta^{-1}(x) \times \lambda^{-1}(y)$ so the local system F^ξ satisfies $F^\xi(xy) = F^\delta(x) \times F^\lambda(y)$ and $F^\xi_{xy,uv} = F^\delta_{x,u} \times F^\lambda_{y,v}$. Now if $\tilde{u} \in ker(\alpha)$ then $\phi(u) \sim 1 \sim \psi(u)$, so $F^\delta_{\phi(u)} = id = F^\lambda_{\psi(u)}$ and hence $F^\xi_u = F^\delta_{\phi(u)} \times F^\lambda_{\psi(u)} = id$. Therefore $\tilde{u} = 1$, so α is injective, completing the proof.

Following Quillen in [14], define an n-dimensional simplicial complex K to be **Cohen-Macaulay** (abbreviated CM) if K is $(n-1)$-connected and $Link_K(s)$ is $(n-k-2)$-connected for each k-dimensional simplex s of K.

(3.5): *Assume K is the flag complex of a geometry Γ over I and K is Cohen-Macaulay. Then each truncation of K is Cohen-Macaulay.*

Proof: Let L be the truncation of K at $J \subseteq I$. Then if I, J have order $n+1, m+1$, respectively, then K, L have dimension n, m, respectively. We proceed by induction on n. If $n = 0$ then K has no nonempty proper truncation, so the induction is anchored.

Let $\iota : L \to K$ be the inclusion map. Then for s a k-simplex of K,

$$\bigcap_{x \in s} \iota^{-1}(st_K(x))$$

is the truncation $\theta(s)$ of $st_K(s)$ at J, as K is the flag complex of Γ. Hence if $\varnothing \neq \tau(s) \cap J$ then $\theta(s)$ is contractible. On the otherhand if $\varnothing = \tau(s) \cap J$ then $\theta(s)$ is the truncation of $Link(s)$ at J, and hence is CM by induction. In particular $\theta(s)$ is $(m-1)$-connected. Thus ι is locally $(m-1)$-connected, (in the language of [8]) so by Theorem 1 in [8], L is $(m-1)$-connected.

Finally if t is a simplex of L then $Link_L(t)$ is the J-truncation of $Link_K(t)$ and hence by induction is CM of dimension $(m-k-1)$. So L is indeed CM.

4. A rank 3 geometry for rank 2 groups

In this section G is a rank 2 group of Lie type with Tits system (G, B, N, S). Let $S = \{s_1, s_2\}$ and $G_i = \langle B, s_i \rangle$ the ith maximal parabolic. Let $G_3 = N$,

$I = \{1, 2, 3\}$, $\mathcal{F} = (G_i : i \in I)$, $\Gamma = \Gamma(G, \mathcal{F})$ the coset geometry defined by $\mathcal{F}$, and $K = \mathcal{C}(G, \mathcal{F})$ the geometric complex define by $\mathcal{F}$. (*cf.* Section 3 in this paper and Sections 3 and 41 in [1]) Observe $G = \langle \mathcal{F} \rangle$ and $G_i = \langle G_{ij}, G_{ik} \rangle$ for all distinct i, j, k from I, where $G_{ij} = G_i \cap G_j$. Thus K is residually connected. (*cf.* 3.2 in [3]) Further from the BN-pair axioms, N is transitive on chambers over N. Thus

(4.1): *K is residually connected, K is the flag complex of Γ, and G is flag transitive on Γ.*

The main result of this section is:

THEOREM 4.2: *K is simply connected.*

The proof involves a short series of reductions. As Γ is residually connected, K is connected. Thus as K is the flag complex of Γ it remains to show Γ is triangulable in the sense of [6].

Let $\mathcal{B}$ be the building of Γ. Write $\mathcal{O}$ for the set of objects of $\mathcal{B}$. Thus $\mathcal{O} = \Gamma_1 \cup \Gamma_2$. Let $\mathcal{A}$ be the apartment set of $\mathcal{B}$ and Σ_N the apartment stabilized by N; then the map $\Sigma_N g \mapsto Ng$, $g \in G$, is a bijection between $\mathcal{A}$ and Γ_3 and we identify $\mathcal{A}$ with Γ_3 via this injection. Thus subject to this convention, incidence in Γ between objects and apartments is inclusion while incidence in Γ between objects is incidence in $\mathcal{B}$. That is Γ is isomorphic to the graph of objects and apartments in the building $\mathcal{B}$.

Write $\mathcal{O}$ for the incidence graph on the set $\mathcal{O}$ of objects, and given $x, y \in \mathcal{O}$ define $d_{\mathcal{O}}(x, y)$ to be the distance from x to y in $\mathcal{O}$. As $\mathcal{B}$ is a linear space, if $x, y \in \mathcal{O}$ with $d_{\mathcal{O}}(x, y) = 2$ then there exists a unique $x + y \in \mathcal{O}(x, y)$. In this case we say x and y are **colinear** and $x + y$ is the unique **line** through x and y.

As K is a geometric complex:

(4.3): *The triangles $xyzx$ of Γ are in one to one correspondance with the flags $\{x, y, z\}$ of Γ, and each such flag consists of a pair of incident objects plus an apartment containing the pair.*

(4.4):

(1) *If $\{x, y\}$ and $\{a, b\}$ are chambers in $\mathcal{B}$ then there exists an apartment Σ with $\{x, y, a, b\} \subseteq \Sigma$.*

(2) *Let $d = diam(\mathcal{O})$. Then $\mathcal{O}$ is a generalized d-gon. In particular if $d_{\mathcal{O}}(x, y) < d$ for some $x, y \in \mathcal{O}$ then there exists a unique geodesic $\rho(x, y)$ from x to y*

in $\mathcal{O}$ and $\rho(x,y) \subseteq \Sigma$ for each apartment Σ containing x and y.

(3) *If $x, y \in \mathcal{O}$ with $d_{\mathcal{O}}(x,y) = d = diam(\mathcal{O})$ then for all $a, b \in \mathcal{O}(x)$, $d_{\mathcal{O}}(y,a) = d_{\mathcal{O}}(y,b) = d-1$ and there exists $\Sigma \in \mathcal{A}$ with $\{x,y,a,b\} \subseteq \Sigma$.*

Proof: Part (1) is one of the building axioms. Let $d = diam(\mathcal{O})$.

It is well known that $\mathcal{O}$ is a generalized d-gon; for example let $x, y \in \Sigma \in \mathcal{A}$ and $z \in \mathcal{O}(x) \cap \Sigma$ with $d_{\mathcal{O}}(y,z) = d_{\mathcal{O}}(y,x)+1$. Then $\{x,z\}$ is a chamber in $\mathcal{B}$ so by 42.3 in [1], each path from x to y in $\mathcal{O}$ of minimal length is in Σ. Hence as Σ is a d-gon that path $\rho(x,y)$ is unique and contained in Σ. Thus (2) is established.

Assume the hypotheses of (3). Then as G is of Lie rank 2, G_x is a maximal parabolic of G and as $d_{\mathcal{O}}(x,y) = d$, $G_x = RG_{xy}$, where R is the unipotent radical of G_x, G_{xy} is a Levi factor of G_x, and G_x is 2-transitive on $\mathcal{O}(x)$. Now R is trivial on $\mathcal{O}(x)$, so G_{xy} is 2-transitive on $\mathcal{O}(x)$. Therefore (3) follows as there exists an apartment Σ containing x and y and as Σ is a d-gon, $\Sigma \cap \mathcal{O}(x)$ is of order 2 and consists of the objects of distance $d-1$ from y in Σ.

(4.5): *xyz is a path of length 2 in Γ with $d_\Gamma(x,z) = 2$ if and only if one of the following holds:*

(1) *$x, z \in \mathcal{O}$ are colinear and $y = x + z$.*

(2) *$x, z \in \mathcal{O}$, $z \notin \mathcal{O}(x)$, and $y \in \mathcal{A}$.*

(3) *$x \in \mathcal{O}$, $z \in \mathcal{A}$ and y is the unique member of $\mathcal{O}(x) \cap z$.*

(4) *$x, z \in \mathcal{A}$ and $y \in x \cap z$.*

Proof: This is straightforward except possibly for the uniqueness statement in (3). But if $y, a \in \mathcal{O}(x) \cap z$ are distinct then $x = y + a \in z$ by 4.4.2, contradicting $d_\Gamma(x,z) = 2$.

Remark: Notice that by 4.5 that if x, y are distinct members of $\mathcal{O}$ then either $y \in \mathcal{O}(x)$ and $d_\Gamma(x,y) = 1$ or $d_\Gamma(x,y) = 2$. ∎

Indeed:

(4.6): *If $x, y \in \mathcal{O}$ with $d_{\mathcal{O}}(x,y) > 2$ then $\Gamma(x,y) = \{\Sigma \in \mathcal{A} : x, y \in \Sigma\}$.*

(4.7): *If $x, y \in \mathcal{O}$ are colinear then each square through x and y in Γ is trivial.*

Proof: Let $p = x_0 \cdots x_4$ be a square with $x_0 = x$ and $x_2 = y$. Then $x + y \in x_i^\perp$ for each i by 4.4.2 and 4.5. Thus p is trivial.

(4.8): *If $x, y \in \mathcal{O}$ with $y \notin x^{\perp}$ then each square through x and y is trivial.*

Proof: By the Remark above, $m = d_{\mathcal{O}}(x,y) \geq d_{\Gamma}(x,y) = 2$ and by 4.6 and 4.7 we may assume $\Gamma(x,y)$ consists of the apartments containing x and y. Let $d = diam(\mathcal{O})$. If $m \neq d$ then by 4.4.2, $\rho(x,y) \subseteq \Sigma$ for all $\Sigma \in \Gamma(x,y)$. Hence $\rho(x,y)$ is a path from x to y in $\Gamma(\Sigma,\theta)$ for each $\Sigma, \theta \in \Gamma(x,y)$ and hence the square $p = x\Sigma y\theta x$ is trivial by 3.4 in [6].

So assume $m = d$. Let $a \in \mathcal{O}(x) \cap \Sigma$ and $b \in \mathcal{O}(x) \cap \theta$. By 4.4.3, there exists $\Xi \in \Gamma(x,y)$ containing a, b. Then by the previous paragraph the squares $a\Sigma y\Xi a$ and $b\Xi y\theta b$ are trivial, so as p is in the closure of these squares and the triangles $xy_iy_{i+1}x$ determined by the path $y_0 \cdots y_4 = \Sigma a\Xi b\theta$ in $\Gamma(x)$, p is trivial.

(4.9): *All squares in Γ are trivial.*

Proof: Let $p = x_0 \cdots x_4$ be a square in Γ. If $x_i, x_{i+2} \in \mathcal{O}$ for some i the lemma holds by 4.8. If $x_0 \in \mathcal{O}$ and $x_2 \in \mathcal{A}$ then by 4.5, $\Gamma(x_0, x_2)$ consists of the unique member of $x_2 \cap \mathcal{O}(x_0)$, contradicting $x_1, x_3 \in \Gamma(x_0, x_2)$. This leaves the case $x_0, x_2 \in \mathcal{A}$. But then $x_1, x_3 \in \mathcal{O}$, a case already handled.

(4.10): *If $p = x_0 \cdots x_n$ is a nontrivial n-gon then $n = 6$ and, translating the origin of p if necessary, $x_i \in \mathcal{O}$ for i even and $x_j \in \mathcal{A}$ for j odd.*

Proof: By 4.3 and 4.9, $n > 4$. Without loss $x_0 \in \mathcal{O}$. If $n \geq 6$ then $d(x_0, x_i) \geq 3$ for $2 < i < n-2$, so by 4.5, $x_i \in \mathcal{A}$. But $x_i \in \mathcal{A}$ implies $x_{i+\epsilon} \in \mathcal{O}$ for $\epsilon = \pm 1$, so $n = 6$, $x_3 \in \mathcal{A}$, and $x_2, x_4 \in \mathcal{O}$. Then by symmetry between x_0 and x_2 and x_4, we have $x_1, x_5 \in \mathcal{A}$.

So take $n = 5$. Now if $x_2, x_3 \in \mathcal{O}$ then $\{x_2, x_3\}$ is a chamber of $\mathcal{B}$, so by 4.4.1, $x_0, x_2, x_3 \in \Sigma \in \mathcal{A}$. But then p is trivial by 1.5 in [7]. Thus we can assume $x_2 \in \mathcal{A}$, so that $x_1, x_3 \in \mathcal{O}$. Now apply the same argument to x_3 in place of x_0 to complete the proof.

We are now in a position to complete the proof of Theorem 4.2. For if Γ is not simply connected then we can choose a nontrivial hexagon $p = x_0 \cdots x_6$ as in 4.10. Let $x = x_0$ and pick p so that $m = \min\{d_{\mathcal{O}}(x, x_i) : i = 2, 4\}$ is minimal. Notice if $d_{\mathcal{O}}(x, x_2) = 2$ then $q = x(x + x_2)x_2 \cdots x_n \sim p$ and by 4.10, q is trivial as $x + x_2 \notin \mathcal{A}$. But then $p \sim 1$. Therefore $m > 2$.

We proceed by induction on m, with the previous paragraph anchoring the induction. Let $y \in \mathcal{O}(x) \cap x_1$ with $d_{\mathcal{O}}(y, x_2) = m - 1$ and Σ an apartment containing y and x_4. Then $r = xy\Sigma x_4x_5x$ is a 5-cycle and hence trivial, while

$d_{\mathcal{O}}(y, x_2) = m - 1$, so by induction on m, $q = yx_1x_2x_3x_4\Sigma y \sim 1$. Hence as p is in the closure of q, r, and the triangle xx_1yx, $p \sim 1$, completing the proof.

5. p-group complexes of a finite group

In this section p is a prime divisor of the order of a finite group G. Let $\Lambda(G) = \Lambda_p(G)$ be the commuting graph on the subgroups of G of order p and $K(G) = K_p(G)$ the clique complex $K(\Lambda_p(G))$ of the graph $\Lambda_p(G)$. Recall the **Brown complex** $\mathcal{S}_p(G)$ of G at p is the order complex of the poset of all nontrivial p-subgroups of G. The **Quillen complex** $\mathcal{A}_p(G)$ is the order complex of the poset of all nontrivial elementary abelian p-subgroups of G.

Write $\mathcal{A}_p^*(G)$ for the simplicial complex whose vertices are the maximal elementary abelian p-subgroups of G and whose simplices are the sets s of vertices such that $\bigcap_{A \in s} A \neq 1$.

Write $\mathcal{B}_p(G)$ for the subcomplex of the Brown complex $\mathcal{S}_p(G)$ consisting of those nontrivial p-subgroups X of G with $X = O_p(N_G(X))$. The complex $\mathcal{B}_p(G)$ is the **Bouc complex** for G at p.

It is known that the complexes $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, $K_p(G)$, $\mathcal{A}_p^*(G)$, and $\mathcal{B}_p(G)$ all have the same homotopy type; we supply proofs of these equivalences in a moment. We usually work with $K(G)$ in this paper. As observed by Quillen in Proposition 2.4 of [14] in the context of the Quillen complex:

(5.1): *(Quillen) If $O_p(G) \neq 1$ then $K_p(G)$ is contractible.*

Proof: Let $Z = \Omega_1(Z(O_p(G)))$ and $\sigma = K(Z)$. Then σ is a simplex of $K(G)$ such that $\sigma \cap s^{\perp} \neq \varnothing$ for each simplex s of $K(G)$, so by 5.1 in [8], $K(G)$ is contractible.

(5.2): *$K_p(G)$, $\mathcal{A}_p(G)$, and $\mathcal{A}_p^*(G)$ have the same homotopy type.*

Proof: This was observed independently by Alperin and in 9.7 of [7]. We fill in details of the proof sketched in [7]; it is a variant of a proof due to Alperin. Let $\mathcal{F}$ be the cover of $\mathcal{A}_p(G)$ consisting of the subcomplexes $F(A) = \mathcal{A}_p(A)$, $A \in \mathcal{A}_p^*(G)$. Then $\mathcal{F}$ is a **contractible cover** of $\mathcal{A}_p(G)$; that is for each $\mathcal{E} \subseteq \mathcal{A}_p^*(G)$, $I(\mathcal{E}) = \bigcap_{A \in \mathcal{E}} F(A)$ is contractible or empty. Namely $I(\mathcal{E}) = \mathcal{A}_p(\bigcap_{A \in \mathcal{E}} A)$, while for $A \in \mathcal{A}_p(G)$, $\mathcal{A}_p(A)$ is contractible as the map $E \mapsto A$ for all $E \in \mathcal{A}_p(A)$ is contiguous to the identity.

As $\mathcal{F}$ is a contractible cover of $\mathcal{A}_p(G)$, $\mathcal{A}_p(G)$ has the same homotopy type as the nerve $N(\mathcal{F})$ of this cover; cf. 4.4.1 of [8]. But of course the map $F : A \mapsto F(A)$

is an isomorphism of $\mathcal{A}_p^*(G)$ with $N(\mathcal{F})$, so $\mathcal{A}_p^*(G)$ and $\mathcal{A}_p(G)$ have the same homotopy type.

Similarly $\mathcal{A}_p^*(G)$ and $K_p(G)$ have the same type. Here we consider the cover $\mathcal{T}$ of $K(G)$ via the sets $T(A) = K(A)$, $A \in \mathcal{A}_p^*(G)$. By 5.1, $\bigcap_{A\in\mathcal{U}} T(A) = K(\bigcap_{A\in\mathcal{U}} A)$ is contractible or empty for each $\mathcal{U} \subseteq \mathcal{A}_p(G)$, so the same argument works.

(5.3): *(Quillen) $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G))$ have the same homotopy type.*

Proof: This is Proposition 2.1 in [14]. As $\mathcal{A}_p(S)$ is contractible for each $S \in \mathcal{S}_p(G)$ by 5.1 and 5.2, the lemma follows from 1.3.

(5.4): *(Bouc) $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ have the same homotopy type.*

Proof: In [10], Bouc supplies only Lemma 1.3 as a proof; we include details here. Let $B = \mathcal{B}_p(G)$ and $S = \mathcal{S}_p(G)$; by 1.3 it suffices to show $S^* \subseteq B$. Let $X \in S$, $T = \mathcal{S}_p(N_G(X))$, and define $\phi : S(> X) \to T(> X)$ by $\phi(Y) = N_Y(X)$. Then ϕ is a map of posets and for $Y \in T(> X)$, $\phi^{-1}(T(\geq Y)) = \{Z > X : Y \leq N_Z(X)\} = S(\geq Y)$ is contractible. So by Proposition 1.6 in [14], ϕ is a homotopy equivalence. Thus $X \in S^*$ if and only if $X \in T^*$, so without loss, $X \trianglelefteq G$.

Now the map $Y \mapsto Y/X$ is an isomorphism of $S(> X)$ with $\mathcal{S}_p(G/X)$. Further if $O_p(G/X) \neq 1$ then $\mathcal{S}_p(G/X)$ is contractible by 5.1–5.3. Thus $S^* \subseteq B$, completing the proof.

(5.5): *Let G be of Lie type of characteristic p and Lie rank l. Let $\mathcal{B}$ be the building of G regarded as a geometric complex. Then*

(1) *$K_p(G)$ and $\mathcal{B}$ have the same homotopy type.*
(2) *$\mathcal{B}$ is Cohen-Macaulay of dimension $l-1$.*
(3) *$K_p(G)$ is $(l-2)$-connected but $H_{l-1}(K_p(G)) \neq 0$. In particular $K_p(G)$ is simply connected if and only if $l \geq 3$.*
(4) *Each truncation of $\mathcal{B}$ is Cohen-Macaulay.*

Proof: As G is of Lie type and characteristic p, the members of $\mathcal{B}_p(G)$ are the unipotent radicals of the proper parabolics of G, and for $Q, P \in \mathcal{B}_p(G)$, $Q \leq P$ if and only if $N_G(P) \leq N_G(Q)$. Thus as $\mathcal{B}$ is the order complex of the poset of proper parabolics, $\mathcal{B}$ and $\mathcal{B}_p(G)$ are isomorphic. So 5.2 through 5.4 imply (1).

Next (2) is well known. For example $\mathcal{B}$ is simply connected by [17], while the homology of $\mathcal{B}$ is known from the Solomon–Tits Theorem [15]. Then (1) and (2) imply (3) while (2) and 3.5 imply (4).

6. Connectivity of p-group complexes

In this section we continue the hypotheses and notation of the previous section. Write $\mathcal{E}_n(G) = \mathcal{E}_n^p(G)$ for the set of elementary abelian p-subgroups of G of rank n. We sometimes view $\mathcal{E}_n^p(G)$ as a graph with A adjacent to B if $[A, B] = 1$; we say $\mathcal{E}_n^p(G)$ is **connected** if this graph is connected. The group $\Gamma_{P,2}^{\circ}(G)$, $P \in Syl_p(G)$, is defined in section 46 of [1], where its relation to the graph $\mathcal{E}_2^p(G)$ is discussed.

(6.1): *$\Lambda_p(G)$ is disconnected if and only if G has a strongly p-embedded subgroup.*

Proof: This is well known; see for example 44.6 in [1].

(6.2): *$\Lambda_p(G)$ is disconnected if and only if either $O_p(G) = 1$ and $m_p(G) = 1$ or $\langle \Lambda_p(G) \rangle / O_{p'}(\langle \Lambda_p(G) \rangle)$ is one of the following:*

(1) *Simple of Lie type of Lie rank 1 and characteristic p.*
(2) *A_{2p} with $p \geq 5$.*
(3) *${}^2G_2(3)$, $L_3(4)$, or M_{11} with $p = 3$.*
(4) *$Aut(Sz(32))$, ${}^2F_4(2)'$, Mc, or $M(22)$ with $p = 5$.*
(5) *J_4 with $p = 11$.*

Proof: This follows from 6.1 and the list of groups with a strongly p-embedded subgroup. See 24.1 in [12] for a proof that the list above is complete.

(6.3): *Let $x \in \Lambda_p(G)$, $H = N_G(x)$, and $P \in Syl_p(H)$. Then the following are equivalent:*

(1) *$\Lambda(x)$ is connected.*
(2) *$Link_{K_p(G)}(x)$ is connected.*
(3) *$\mathcal{E}_2^p(H)$ is connected.*
(4) *Either $|\mathcal{E}_2^p(H)| = 1$ or*
  (a) *For all $E \in \mathcal{E}_2^p(H)$, $m_p(C_H(E)) > 2$, and*
  (b) *$H = \Gamma_{P,2}^{\circ}(H)$.*

Proof: First as $K(G)$ is the clique complex of Λ, $\Lambda(x) = Link(x)$ for each $x \in \Lambda$. Next given a path $q = y_0 \cdots y_n$ in $\Lambda(x)$, observe $\rho(q) = (xy_0) \cdots (xy_n)$ is a path in $\mathcal{E}_2(H)$. Conversely given a path $r = E_0 \cdots E_m$ in $\mathcal{E}_2(H)$, notice $\tau(r) = z_0 \cdots z_m$ is a path in $\Lambda(x)$, where z_i is any member of $\Lambda(E_i) - \{x\}$. In particular (1) is equivalent to (2) and (3).

Next if $|\mathcal{E}_2(H)| = 1$ then clearly $\mathcal{E}_2(H)$ is connected. On the otherhand if $E \in \mathcal{E}_2(H)$ with $m_p(C_H(E)) = 2$ then E is an isolated vertex in $\mathcal{E}_2(H)$, so if $\mathcal{E}_2(H)$ is connected then $\{E\} = \mathcal{E}_2(H)$. Hence in proving the equivalence of (3) and (4) we may assume condition (4a) holds. Now 46.7.3 in [1] completes the proof.

(6.4): *Let $x \in \Lambda_p(G)$, $R \trianglelefteq G$ with $R \leq O_{p'}(G)$, and $\bar{G} = G/R$. Then*

(1) *If $\Lambda(x)$ is connected then $\Lambda(\bar{x})$ is connected.*

(2) *If $K_p(G)$ is simply connected then $K_p(\bar{G})$ is simply connected.*

Proof: If $q = y_0 \cdots y_n$ is a path in $\Lambda(x)$ then $\bar{q} = \bar{y}_0 \cdots \bar{y}_n$ is path in $\Lambda(\bar{x})$, so (1) holds.

Assume $K = K_p(G)$ is simply connected, $m_p(G) > 1$, and let $D = K_p(\bar{G})$. Now $f : K \to D$ defined by $f(x) = \bar{x}$ is a simplical map. Further each k-simplex of D is of the form $\bar{s} = \{\bar{x}_0, \ldots, \bar{x}_k\}$ for some k-simplex $s = \{x_0, \ldots, x_k\}$ of K and $f^{-1}(\bar{s}^\perp) = K_p(C_G(s)R)$, so $f^{-1}(\bar{s}^\perp) \neq \varnothing$ and if $k = 0$ then $f^{-1}(\bar{s}^\perp)$ is connected by 6.1 and 6.2. Namely $1 \neq x_0 \leq O_{p',p}(C_G(x_0)R)$ so we have connectivity unless $m_p(C_G(x_0)) = 1$. But in that event $m_p(G) = 1$, so as K is connected, $x_0 \leq O_p(G)$, and then $K_p(C_G(x_0)R)$ is connected.

We have shown f to be locally connected in the language of [8]; thus (2) follows from Theorem 2 of [8].

(6.5): *Assume the Conjecture holds in all applicable sections of G and assume for all $x \in \Lambda_p(G)$ that $\Lambda(x)$ is connected. Then if $K_p(G/O_{p'}(G))$ is simply connected, so is $K_p(G)$.*

Proof: Assume $K(G/O_{p'}(G))$ is simply connected; then it remains to show $K(G)$ is simply connected. Proceeding by induction on the order of G, take G to be a minimal counter example. In particular $G = \langle \Lambda_p(G) \rangle$. Let H be a minimal normal subgroup of G contained in $O_{p'}(G)$.

Suppose $G = AH$ for some $A \in \mathcal{A}_p(G)$. Then by minimality of H, either H is a q-group for some prime $q \neq p$ and A is irreducible on H, or H is the direct product of the A-conjugates of a nonabelian simple p'-group L. In the second case the lemma holds by the Conjecture. In the first as A is irreducible on H, $O_p(G) \neq 1$, so $K(G)$ is contractible. Thus we may assume $G \neq AH$ for $A \in \mathcal{A}(G)$.

Let $\bar{G} = G/H$. Then by 6.4, our hypotheses are inherited by $K(\bar{G})$, so by induction on the order of G, $K(\bar{G})$ is simply connected. Hence by 5.2, $Q = \mathcal{A}_p(\bar{G})$

is simply connected and it remains to show $P = \mathcal{A}_p(G)$ is simply connected. We will do so by appealing to Lemma 1.4.

Let $f : P \to Q$ be the projection $f : A \mapsto \bar{A}$. Then f is a map of posets. Notice for $A \in P$, $\bar{A}$ is of height h in Q if and only if $m_p(A) = h + 1$. By induction on the order of G, $f^{-1}(Q(\leq A)) = \mathcal{A}_p(AH)$ is simply connected if $m_p(A) > 2$, since then $\Lambda(x)$ is connected for each $x \in \Lambda(AH)$ by 6.3. Similarly if $m_p(A) = 2$ then $\mathcal{A}_p(AH)$ is connected by 6.2. That is hypothesis (1) of Lemma 1.4 is satisfied.

So it remains to verify hypothesis (2) of Lemma 1.4. Now for $A \in P$ of p-rank 1, $\Lambda(A)$ is connected by hypothesis, so $P(> A)$ and hence also $Q(> \bar{A})$ is connected. Finally let $m_p(A) = 2$. Then $Q(> \bar{A}) \neq \varnothing$ unless $m_p(C_G(A)) = 2$. But in that case as $\Lambda(x)$ is connected for $x < A$ of p-rank 1, 6.3 says $\{A\} = \mathcal{E}_2^p(C_G(x))$. Hence $H = \Gamma_{A,1}(H) \leq N_G(A)$, so $A = \langle \Lambda(AH) \rangle$, and hence $\mathcal{A}_p(A) = \mathcal{A}_p(AH)$ is simply connected by 5.1. Thus we have verified hypothesis (2) of Lemma 1.4, completing the proof.

(6.6): *If $O_p(G) = 1$, $\Lambda_p(G)$ is connected, and $m_p(G) = 2$, then $H_1(K_p(G)) \neq 0$.*

Proof: If $\mathcal{A}(G)$ is a tree then as $\mathcal{A}(G)$ is finite and bipartite, G has a fixed point on $\mathcal{A}(G)$, contradicting $O_p(G) = 1$. So there exists an r-gon q in $\mathcal{A}(G)$. By 3.1.4 and 3.1.5 in [8], q determines a cycle $0 \neq \psi(q) \in Z_1(\mathcal{A}(G))$, so as $dim(\mathcal{A}(G)) = 1$, $H_1(\mathcal{A}(G)) \neq 0$.

The next lemma is essentially Proposition 2.6 in Quillen [14].

(6.7): *$K_p(G \times H)$ has the same homotopy type as $K_p(G) \vee K_p(H)$.*

Proof: Let $L = K(G \times H)$ and $D = K(G) \vee K(H)$. Define $\iota : D \to L$ by $\iota(x) = x$ and $\phi : L \to D$ by $\phi(a) = \phi_G(a)$ if $\phi_G(a) \neq 1$, and $\phi(a) = \phi_H(a)$ if $\phi_G(a) = 1$, where $\phi_Y : G \times H \to Y$ is the projection. Then ι and ϕ are simplicial maps with $\phi \circ \iota = id_D$. Further $a^{\perp} \subseteq \iota(\phi(a))^{\perp}$ for all $a \in L$, so by 9.3 in [7], $\iota \circ \phi$ is homotopic to id_L.

(6.8): *Let $H \leq G$ such that $K_p(C_H(E))$ is $(n-j+1)$-connected for all $E \in \mathcal{E}_j^p(G)$ and all $1 \leq j \leq n+2$. Let $\iota : K_p(H) \to K_p(G)$ be inclusion. Then*

(1) *$\iota : K_p(H)^n \to K_p(G)^n$ is a homotopy equivalence of n skeletons.*

(2) *$K_p(H)$ is n-connected if and only if $K_p(G)$ is n-connected.*

(3) *If $n \geq 1$ then $\iota_* : \pi_1(K_p(H)) \to \pi_1(K_p(G)$ is an isomorphism.*

Proof: Let $s = (x_0, \ldots, x_k)$ be a k-simplex in $K(G)$ and $E = \langle x_0, \ldots, x_k \rangle$.

Then $E \in \mathcal{E}^p_j(G)$ with $j \leq k+1$ and $\iota^{-1}(st_{K(G)}(s)) = K(C_H(E))$. So $n - k \leq n - j + 1$ and hence by hypothesis, ι is locally n-connected in the sense of [8]. Then Theorem 1 in [8] completes the proof.

(6.9): *Let $H \trianglelefteq G$ such that $K_p(C_H(x))$ is connected for all $x \in \Lambda_p(G) - \Lambda_p(H)$. Assume $K_p(H)$ is simply connected. Then $K_p(G)$ is simply connected.*

Proof: Let $\iota : K_p(H) \to K_p(G)$ be inclusion. As $H \trianglelefteq G$, for all $E \in \mathcal{E}^p(G)$, $\Lambda(C_H(E)) \neq \varnothing$, so $\iota^{-1}(st_{K(G)}(s)) \neq \varnothing$ for all simplices s of $K(G)$. Further for $x \in \Lambda(G)$, $\iota^{-1}(st_{K(G)}(x)) = K(C_H(x))$. Now $K_p(C_H(x))$ is connected by hypothesis if $x \not\leq H$, while if $x \leq H$ the same is true by 5.1. Thus ι is locally connected in the sense of [8], so Theorem 2 in [8] completes the proof.

(6.10): *Let $\bar{G} = G/O_{p'}(G)$ and $1 \neq H \trianglelefteq G$. Assume the Conjecture holds in proper sections of G and*

(1) *$K_p(\bar{H})$ is simply connected.*
(2) *For all $x \in \Lambda_p(G) - \Lambda_p(H)$, $K_p(C_H(x))$ is connected.*
(3) *For all $x \in \Lambda_p(H)$, $\Lambda(x)$ is connected.*
Then $K_p(G)$ is simply connected.

Proof: Let $x \in \Lambda_p(G) - \Lambda_p(H)$ and $y \in \Lambda(x)$. Then as $H \trianglelefteq G$, $\Lambda(C_H(xy)) \neq \varnothing$, so as $\Lambda_p(C_H(x))$ is connected, $\Lambda(x)$ is connected. By hypothesis, $\Lambda(x)$ is connected if $x \in \Lambda(H)$. So by 6.5, we may assume $O_{p'}(G) = 1$. Now 6.9 completes the proof.

7. p^2-subgroups whose centralizer is of p-rank 2

In this section p is a prime and G a finite group. Let $\Lambda(G) = \Lambda_p(G)$ be the commuting graph on the subgroups of G of order p. Assume $m_p(G) > 2$ but $E_{p^2} \cong A \leq G$ with $m_p(C_G(A))) = 2$.

(7.1): *Let $A \leq P \in Syl_p(G)$ and $Z = \Omega_1(Z(P))$. Then*

(1) $A = \langle \Lambda_p(C_G(A)) \rangle$.
(2) $Z \leq A$.
(3) $|Z| = p$.
(4) *Either*
(a) *$m_p(C_G(a)) = 2$ for each $a \in A - Z$, or*
(b) *$A \in Syl_p(C_G(A))$ and $\langle \Lambda_p(N_G(A)) \rangle$ induces $SL_2(p)$ on A.*

Proof: As $m_p(C_G(A)) = 2 = m_p(A)$, $\Lambda(A) = \Lambda(C_G(A))$, so $Z \leq A$. If $Z = A$ then $2 = m_p(C_G(A)) = m_p(G) > 2$, a contradiction. So (1)–(3) hold and it remains to prove (4).

As $m_p(P) = m_p(G) > 2$, there is $Y \in \Lambda(N_P(A)) - \Lambda(A)$. Similarly we may assume $Z \neq X \in \Lambda(A)$ with $m_p(N_G(X)) > 2$, so there is $U \in \Lambda(N_{N_G(X)}(A)) - \Lambda(A)$. Then $\langle Y, U\rangle$ induces $SL_2(p)$ on A, so it remains to prove $A \in Syl_p(C_G(A))$.

We may assume $R = C_P(A) \in Syl_p(C_G(A))$. Let $M = N_G(A)$ and $H = C_G(A)$. Then by a Frattini argument $M = HN_M(R)$, so $N_M(R)$ is transitive on $A^{\#}$.

Next as $m_p(P) > 2$ we may choose $ZY \trianglelefteq P$. (*cf.* Exercise 8.4 in [1].) As $Y \leq N_M(R)$ and $N_M(R)$ is transitive on $A^{\#}$ we may take $K = \langle Y, U\rangle \leq N_M(R)$ and $U \in Y^K$. As $ZY \trianglelefteq P$, $|P : C_P(Y)| = p$ so $R = C_R(Y)A$. Then as $U \in Y^K$ also $R = C_R(U)A$, so $R = A \times C_R(K)$. Hence by (1), $R = A$, completing the proof of (4).

In the remainder of this section Z and P are as in lemma 7.1.

(7.2): *Assume $O_{p'}(G) = O_p(G) = 1$. Then one of the following holds:*

(1) *G is almost simple, $A \leq F^*(G)$, and $\langle\Lambda_p(N_G(A))\rangle$ induces $SL_2(p)$ on A.*
(2) *G is almost simple and Z is the unique $X \in \Lambda_p(A)$ with $m_p(N_G(X)) > 2$.*
(3) *p is odd, G has p components $\{L_i : 1 \leq i \leq p\}$, these components are permuted regularly by each $X \in \Lambda(A) - \{Z\}$, $m_p(L_i) = 1$, and $\langle\Lambda(N_G(X))\rangle = X \times L$ with $L \cong L_1$.*

Proof: Let $H = F^*(G)$. Then $H = L_1 \times \cdots \times L_n$ is the direct product of the components of G. Further $Z \cap H \neq 1$, so $Z \leq H$.

Notice if 7.1.4.b holds then $A = \langle Z^{N_G(A)}\rangle \leq H$. More generally assume $A \leq H$. Then $2 = m_p(C_G(A)) \geq \sum_i m_i$, where $m_i = 1$ if $m_p(L_i) = 1$ and $m_i \geq 2$ otherwise. We conclude either H is simple or $n = 2$ and $m_p(L_i) = 1$ for each i. But in the later case $p \neq 2$ and hence $A = \Omega_1(P \cap H) \leq Z$, contradicting 7.1.3.

Thus (1) or (2) holds in this case. Hence we may assume $A \not\leq H$ and $m_p(N_G(X)) = 2$ for $Z \neq X \in \Lambda(A)$ and $n > 1$. Now if X acts on some product $K = \prod_i L_i$ of components, then there exists $Y \in \Lambda(C_K(A))$. So as $A = \langle\Lambda(C_G(A))\rangle$, $Y \leq A \cap H = Z$. It follows that X is regular on the components of G, so $n = p$. Also $N_H(X) = L \cong L_1$, so as $m_p(N_G(X)) = 2$, $m_p(L_1) = m_p(L) = 1$. Therefore p is odd and (3) holds.

(7.3): *Assume $p = 2$. Then*

(1) *$A \cap O^2(G) \leq Z$.*

(2) *$m_2(C_G(a)) = 2$ for each $a \in A - Z$.*

(3) *One of the following holds:*

(a) *$Z \leq O_{2',2}(G)$.*

(b) *$F^*(G/O(G)) \cong L_2(q^2)$ with q odd, and A induces inner-diagonal automorphisms.*

(c) *$O^{2'}(G/O(G))$ is $L_3(4)$ extended by a graph-field automorphism.*

Proof: If (2) fails then by 7.1.4, $A \in Syl_2(C_G(A))$. But then by a lemma of Suzuki (*cf.* Exercise 8.6 in [1]) P is dihedral or semidihedral, contradicting $m_2(G) > 2$. Hence (2) holds.

Let $a \in A - Z$. We observed during the proof of 7.1 that there is $E_{p^2} \cong E \trianglelefteq P$. Then by (2), $a^G \cap C_P(E) = \varnothing$, so (1) holds by Thompson transfer.

To prove (3) we may take $O_\infty(G) = 1$. Then G is almost simple by 7.2; let $F^*(G) = L$. Further a induces an outer automorphism on L such that $m_2(C_L(a)) = 1$. Therefore $SCN_3(P) = \varnothing$ so L is described in the Main Theorem of [11] and we conclude that either $L \cong L_2(r)$ with r odd, and a induces a diagonal automorphism, or $L \cong L_3(4)$ and a induces a graph-field automorphism. In the first case as $m_2(G) > 2$, r is a square, so (3b) holds. In the second as $m_2(C_G(a)) = 2$, $O^{2'}(G) = AL$ and (3c) holds.

(7.4): *Let $p = 2$ and $O(G) = 1 \neq O_2(G)$. Then one of the following holds:*

(1) *$F^*(G) = O_2(G)$.*

(2) *$F^*(G) = L_1 \times L_1^a$ with $L_1^a \neq L_1 \cong SL_2(q)$, q odd, or $A_7/\mathbf{Z}_2$.*

(3) *$F^*(G) = L * K$ where AK is dihedral, semidihedral, or $SL_2(r)$, r odd, extended by a diagonal outer automorphism, and either $[L, a] = 1$ and $L \cong SL_2(q)$, q odd, or $A_7/\mathbf{Z}_2$, or a induces an outer automorphism on $L \cong SL_2(q)$ or $Sp_4(q)$, q odd, $A_n/\mathbf{Z}_2$, $n = 7, 8, 9$, or $SL_4^\epsilon(q)$, $q \equiv -\epsilon$ mod 4.*

(4) *$E(G) \cong SL_2(q)$, q odd, and a induces a diagonal outer automorphism on $E(G)$.*

Proof: Let $Q = O_2(G)$ and assume $E(G) \neq 1$. Let $a \in A - Z$ and K an A-invariant subnormal subgroup of $F^*(G)$. Then $Z \leq K$. In particular $Z \leq Q$, so even $Z \leq O_2(K)$.

If $A \in Syl_2(C_{AK}(A))$ then by Suzuki's Lemma, KA has dihedral or semidihedral Sylow 2-subgroups. Hence as $Z \leq O_2(K)$, either $K \leq Q$ or $K \cong SL_2(q)$

for some odd q, and a induces a diagonal outer automorphism on K. In particular if K_1 and K_2 are two such distinct subgroups with $[K_1, K_2] = 1$ and $|K_i| > 4$, then a inverts an element k_i of order 4 in K_i. But then $E_4 \cong \langle k_1k_2, Z\rangle \leq C_G(A)$, so $A = \langle k_1k_2, Z\rangle \leq K_1K_2$, impossible as a induces an outer automorphism on K_i.

On the otherhand if $A \notin Syl_2(C_{AK}(A))$ then $C_K(A)$ contains an element of order 4, and if K_1 and K_2 are two such commuting subgroups with elements k_i of order 4 then again $A = \langle k_1k_2, Z\rangle \leq K_1K_2$.

Next if $J \neq J^a$ for some component J of G then as $E(C_{JJ^a}(a)) \cong J/I$ where $I = \{j \in J : j^a = j^{-1}\}$ and $m_2(C_{JJ^a}(a)) = 1$, $J \cap J^a = 1$ and $m_2(J) = 1$. Further by the previous paragraph, if we let $K = JJ^a$ then $AC_G(K)$ has dihedral or semidihedral Sylow 2-groups, so as $E_4 \cong Z(K) \leq Z(C_G(K))$, $Z(K) = C_G(K)$. So (2) holds.

So assume A fixes each component J. Then $Z \leq J$. Now as $m_2(C_G(a)) = 2$, $SCN_3(P) = \varnothing$. Hence J is described in 1.2.

Now if a induces an inner automorphism on J then $a = cj$, $c \in C_G(J)$, $j \in J$ of order 1,2, or 4. If j is an involution then $A = \Lambda(C_G(A)) = Z\langle j\rangle \leq J$, contradicting 7.3.1. On the otherhand if $|j| = 4$ then $C_{AJ}(a) = \langle a\rangle \times C_J(j)$, so $m_2(C_J(j)) = 1$, which by Suzuki's Lemma applied to J/Z forces $J \cong SL_2(q)$ or $A_7/\mathbf{Z}_2$. As j is inverted by an element of J of order 4 and $A = \Lambda(C_G(A))$, c is not inverted by an element of $C_G(J)$ of order 4 and $C_G(J\langle c\rangle)$ is cyclic. It follows from Suzuki's Lemma that $C_G(J)$ is cyclic or dihedral. But then $A \trianglelefteq T \in Syl_2(G)$, so as $m_2(T) > 2$, $m_2(C_T(A)) > 2$, a contradiction. Finally if $[a, J] = 1$ then by paragraph two, (3) holds.

Thus a induces an outer automorphism on each component J of G. Then by paragraphs two and three, (3) or (4) holds.

(7.5): *Let p be odd and $O_{p'}(G) = 1 \neq O_p(G)$. Then either*

(1) $F^*(G) = O_p(G)$

(2) $F^*(G) = O_p(G)L$ *where* $L = E(G)$ *is quasisimple,* $O_p(G)$ *is cyclic,* $A = ZX$ *with* $Z = A \cap F^*(G) \leq O_p(G) \cap L$, $L \cong SL^\epsilon_p(q)$ *with* $q \equiv \epsilon \mod p$, *and* X *induces diagonal outer automorphisms on* L.

Proof: Let $Q = O_p(G)$ and assume $Q \neq F^*(G)$. Then $L = E(G) \neq 1$ and in particular $\Lambda(L) \nsubseteq Q$. As $A = \langle\Lambda(C_G(A))\rangle$, $A \cap Z(Q) \neq 1$. As $\Lambda(L) \nsubseteq Q$, $A \nleq Q$, so $A \cap Q = A \cap Z(Q) = Z$. Let $Z \neq X \in \Lambda(A)$.

If X moves some component J of G then $\langle J^X \rangle \cap N_G(X) = I$ is a homomorphic image of J. As $p \neq 2$, $\Lambda(I) \not\subseteq Q$. But $\Lambda(I) \subseteq \Lambda(C_G(A)) \subseteq A$, a contradiction. So A fixes each component J of G. Thus $\varnothing \neq \Lambda(C_J(A)) \subseteq A$, so either $Z \leq J$ or $m_p(J) = 1$ and we may take $X = A \cap J$. But in the latter case $m_p(G) = m_p(C_G(A)) = 2$, a contradiction. Now X acts on some $E_{p^2} \cong B_J \leq J$. So if K is a second component then as $[B_K B_J, A] \leq Z$, $m_p(C_G(A) \cap B_K B_J) > 2$, a contradiction. Hence L is quasimple. Similarly Q is cyclic. Finally Lemma 29.1 in [12] identifies L and completes the proof.

(7.6): *Let p be odd, $O_{p'}(G) = 1$, $F^*(G) = L$ simple, and $a \in A^\#$ with $2 = m_p(C_G(a))$. Then one of the following holds:*

1. *$L \cong L_p^\epsilon(q)$ with q odd, $q \equiv \epsilon \mod p$, and a induces an outer diagonal automorphism on L.*
2. *$L \cong L_2(p^p)$ and a induces a field automorphism on L.*
3. *$p \geq 5$, $a \in L$, and $L \cong PSp_4(p)$ or $G_2(p)$.*

Proof: See 29.1 in [12]. While Co_1 with $p = 5$ appears in the conclusion of 29.1 it is not a real example as can be seen from the discussion in the proof of 7.7 below.

(7.7): *Let p be odd, $O_{p'}(G) = 1$, $F^*(G) = L$ simple, and $N_G(A)$ transitive on $A^\#$. Then one of the following holds:*

1. *$L \cong A_n$ with $n = p^2 + r$, $0 \leq r < p$, and A has a regular orbit on the n-set of L.*
2. *$L \cong Co_1$ and $p = 5$.*
3. *$L \cong L_p^\epsilon(q)$ with $q \equiv \epsilon \mod p$, or $p = 3$ and $L \cong G_2(q)$, or ${}^3D_4(q)$. Further if $p = 3$ then some element of order 3 induces a graph or field automorphism on L.*

Proof: Notice that as $N_G(A)$ is transitive on $A^\#$ and $Z \leq L$, we have $A \leq L$. By 7.1, $Z(P)$ is cyclic. Thus if L is of Lie type in characteristic p then L is defined over $GF(p)$ and Z is the center of a long root group. Indeed as $m_p(G) > 2$, $F^*(C_G(Z)) = Q$ is extraspecial. Then as $N_G(A)$ is transitive on $A^\#$, A is determined up to conjugacy in $Aut(L)$, $A \leq Q$, and L is not unitary or symplectic. Hence as $m_p(C_G(A)) = 2$, Q has width 1. But then $m_p(G) = 2$, a contradiction.

Suppose $L = A_n$. Then as $Z(P)$ is cyclic, $n = p^e + r$, $0 \leq r < p$, and Z has p^{e-1} orbits of length p. Then as $A^\#$ is fused in L, A has p^{e-2} orbits of length p^2

and r fixed points. Finally as $A = \langle \Lambda(C_G(A)) \rangle$, we conclude $e = 2$ and (1) holds.

If $p = 3$ then as $m_3(C_G(A)) = 2$, $SCN_4(3) = \varnothing$ and hence by Exercise 8.11 in [1], $m_3(G) = 3$.

Suppose L is sporadic. Then as $m_p(G) \geq 3$ with equality when $p = 3$, 10.6 in [12] says either $p = 3$ and $L \cong J_3$, or $p = 5$ and $L \cong Co_1$, Ly, F_5, F_2, or F_1, or $p = 7$ and $L \cong F_1$. In the first case $Z(P)$ is noncyclic, a contradiction.

If $L = Co_1$ then one can calculate that there exists $A \cong E_{5^2}$ admitting the action of $SL_2(5)$, but that A is not fused into $J(P) \cong E_{p^3}$. On the otherhand from page 49 of [12], $m_5(C_G(x)) \geq 3$ for each element x of order 5 in G, so case (c) of Lemma 29.1 of [12] should not appear.

In the remaining cases we check A does not exist from the list of maximal p-local subgroups of G.

Thus we have reduced to the case L of Lie type over $GF(q)$ with q prime to p. Now if L is classical and p is prime to the order d of the center of the universal Chevalley group $\tilde{L}$ of L, then A is contained in an elementary abelian p-subgroup of L of maximal rank and L is transitive on such subgroups. (*cf.* 10.2.4 in [12]) Similarly if L is exceptional and p is prime to the order w of the Weyl group of the algebraic group of L then $P \cap L$ is abelian, so the same conclusion holds. (*cf.* 10.1.3 in [12]) So p divides d or w in the respective case, unless possibly $m_p(L) = 2$. However this last case is impossible, for otherwise by our restriction on p, some element x of order p in G induces a field automorphism on L. But $m_p(C_L(x)) = m_p(L)$, so x centralized a conjugate of A by transitivity of L on its elementary abelian p-subgroups of maximal rank.

In particular if L is exceptional then either $p = 3$ or $p = 5$ and L is of type E_n or ${}^2E_6$, or $p = 7$ and L is of type E_7 or E_8.

Next if $p = 3$ then $m_3(G) = 3$ by an earlier observation. Hence $m_3(L) \leq 3$, so by 10.6 in [12], L has Lie rank 1 or 2 and $m_3(L) \leq 2$. Thus some element of order 3 in G induces an outer automorphism on L and (3) holds.

So $p > 3$. Similarly if L is classical then as p divides d, $L \cong L_n^{\epsilon}(q)$ with $n \equiv q - \epsilon \equiv 0 \mod p$. Then as $SCN_{p+1}(P) = \varnothing$, $n = p$, so (3) holds.

We have reduced to the case L is exceptional of type E_n or ${}^2E_6$ and $p = 5$ or 7. Here we can repeat the argument in paragraphs two and three on page 400 of [12] to complete the proof.

8. p-locals with a proper 2-generated p-core

In this section p is a prime and G a finite group. Let $\Lambda(G) = \Lambda_p(G)$ be the commuting graph on the subgroups of G of order p. Let $X \in \Lambda(G)$, $H = N_G(X)$, $C_X = \langle \Lambda(H) \rangle$, $\bar{H} = H/O_{p'}(H)$, and $P \in Syl_p(C_X)$. Assume also that:
(i) $\Gamma^{\circ}_{P,2}(H) \neq H$.
(ii) $m_p(C_G(A)) > 2$ for all $A \in \mathcal{E}_2^p(G)$.
(iii) $O_{p'}(G) = 1$.

Recall the notation $\mathcal{E}_k^p(G)$ and $\Gamma^{\circ}_{P,2}(G)$ defined in section 6.

(8.1): *If $A \in \mathcal{A}_p(H)$ with $AO_{p'}(H) \trianglelefteq H$ then $A = X$.*

Proof: Assume $A \neq X$. Replacing A by AX, we may assume $X \leq A$. Then A is noncyclic. By (ii), $m_p(P) > 2$ so by 46.2 in [1], $N_H(A) \leq \Gamma^{\circ}_{P,2}(H)$ and of course $O_{p'}(H) \leq \Gamma^{\circ}_{P,2}(H)$, while by a Frattini argument, $H = O_{p'}(H)N_H(A)$. This contradicts (i).

(8.2): *Assume $F^*(\bar{H}) \neq O_p(\bar{H})$. Then one of the folowing holds:*

(1) *$\bar{C}_X = \bar{X} \times \bar{K}_X$, where $\bar{K}_X$ has a strongly p-embedded subgroup with $1 < m_p(\bar{K}_X)$.*
(2) *$p = 2$ and $\bar{H} \cong A_9/\mathbf{Z}_2$ is quasisimple.*
(3) *$p = 2$ and $C_X = K_X C_{C_X}(\bar{K}_X)$ where $\bar{K}_X \cong Sz(8)/\mathbf{Z}_2$, $SL_2(5)$, or $SL_2(5) * SL_2(5)$ is perfect and $m_2(C_{C_X}(\bar{K}_X)) = 1$.*

Proof: Without loss $G = C_X$. Let $M = F^*(H)$.

Observe first that $m_p(C_{PM}(A)) > 2$ for each $A \in \mathcal{E}_2^p(M)$ by 7.3.1 and 7.5. In particular we have $\Gamma_{P\cap M,2}(H) \leq M\Gamma^{\circ}_{P,2}(H)$.

Suppose next that $1 \neq J \trianglelefteq E(H)$ is P-invariant, $J \leq \Gamma^{\circ}_{P,2}(H)$, and $m_p(XJ) > 1$. Then there is $U \in \mathcal{E}_2^p(P \cap XJ)$ with $m(C_P(U)) > 2$, so $N_G(U) \leq \Gamma^{\circ}_{P,2}(H)$. So $M \leq JC_H(U) \leq \Gamma^{\circ}_{P,2}(H)$. But then by a Frattini argument and the previous paragraph, $H = M\Gamma_{P\cap M,2}(H) \leq \Gamma^{\circ}_{P,2}(H)$, contrary to (i).

Thus no such J exists. Now if L is a component of H with $m_p(L) > 1$ if $p = 2$ and $X \not\leq L$, then $m_p(XL) > 1$, so applying the previous paragraph to $J = \langle L^P \rangle$, we conclude $J \not\leq \Gamma^{\circ}_{P,2}(H)$ and hence $L \not\leq \Gamma^{\circ}_{N_P(L),2}(H)$. Therefore $m(C_P(L)) \leq 1$ so $P \leq N_H(L)$ and if p is odd then $L = E(H)$.

Now if p is odd then as $L \not\leq \Gamma^{\circ}_{P,2}(H)$, 24.9 in [12] says L has a strongly p-embedded subgroup. In particular $X \not\leq L$ and as $C_P(L)$ is cyclic and $H = \langle \Lambda(H) \rangle$, $X = C_H(L)$ and H/X has a strongly p-embedded subgroup. Therefore

H/X is described in 6.2. Indeed as $L \cap X = 1$ and $H = \langle \Lambda(H) \rangle$, H splits over X, so $H = X \times K_X$ where $K_X \cong H/X$, and (1) holds.

So $p = 2$. Suppose $\Gamma_{P,2}(H) \leq \Gamma^{\circ}_{P,2}(H)$. Then Theorem 1 of [2] says that either $H \cong A_9/\mathbf{Z}_2$, or $H = KC_H(K)$, where $m_2(C_H(K)) = 1$ and K is simple with a strongly embedded subgroup or $K \cong Sz(8)/\mathbf{Z}_2$, $SL_2(5)$, or $SL_2(5) * SL_2(5)$. In the first case (2) holds and in the last three cases (3) holds. If K has a strongly embedded subgroup, then as H is generated by involutions, $H = K \times C_H(K)$ with $C_H(K)$ generated by involutions, so as $m_2(C_H(K)) = 1$, $X = C_H(K)$ and (1) holds.

Finally assume $\Gamma_{P,2}(H) \not\leq \Gamma^{\circ}_{P,2}(H)$. Then $m_2(C_P(A)) = 2$ for some $E_4 \cong A \leq P$. Then $Z(P)$ is cyclic and by 46.2 in [1], $SCN_3(P) = \varnothing$. Thus $X \leq L$ for each component L of H and L is described in 1.2.1. Suppose $m(L) > 1$. We have seen $P \leq N_H(L)$ and $L \not\leq \Gamma^{\circ}_{P,2}(H)$ so if $\Gamma^{\circ}_{P\cap L,2}(L) = \Gamma_{P\cap L,2}(L)$ then by Theorem 1 in [2], $L \cong A_9/\mathbf{Z}_2$. Now by (ii), $H = L$ and (2) holds. Thus $\Gamma^{\circ}_{P\cap L,2}(L) \neq \Gamma_{P\cap L,2}(L)$ so by 1.2.2, $L \cong Sp_4(q)$ for some odd q. Now for $A \in \mathcal{E}^2_3(P)$ with $A \cap L \in \mathcal{E}^2_2(L)$, we find $L = \Gamma_{A,2}(L) \leq \Gamma^{\circ}_{P,2}(H)$, contrary to an earlier reduction.

Thus each component of H is of 2-rank 1. Now if L_1 and L_2 are distinct components and $Q_8 \cong P_i \leq L_i$ with $P_i \leq P$, then $\Gamma_{P_1P_2,2}(H) \leq \Gamma^{\circ}_{P,2}$, so as $L_1L_2 \not\leq \Gamma^{\circ}_{P,2}$, $L_i \cong SL_2(5)$ for $i = 1$ or 2 and $L_{3-i} \leq \Gamma^{\circ}_{P,2}$ if L_{3-i} is not $SL_2(5)$. We conclude that (3) holds.

In the remainder of this section if $F^*(\bar{H}) \neq O_p(\bar{H})$ then K_X is the preimage in $N_G(X)$ of the subgroup $\bar{K}_X$ of lemma 8.2, with $K_X = C_X$ in 8.2.2. Further set $L_X = O^{p'}(K_X^{\infty})$.

(8.3): *Assume $F^*(\bar{H}) = O_p(\bar{H})$. Then one of the following holds:*

(1) *$p = 3$ and $\bar{C}_X$ is the split extension of 3^{1+2} by $GL_2(3)$.*

(2) *$p = 2$ and $\bar{C}_X \cong GL_2(3) * D_{2^n}$ or $GL_2(3)YD_{2^n}$, $n \geq 3$, $D_{10}/(Q_8 * D_8)$, $A_5/(Q_8 * D_8)$, or $S_5/(Q_8 * D_8)$.*

Proof: Again we may take $O_{p'}(H) = 1$ and $H = \langle \Lambda(H) \rangle$. Let $Q = O_p(H)$. If $m_p(Q) > 2$ then $H = \Gamma^{\circ}_{Q,2}(H) \leq \Gamma^{\circ}_{P,2}(H)$, so that (i) supplies a contradiction. Therefore $m_p(Q) \leq 2$. In particular $P \neq Q$, so $Aut(Q)$ is not p-closed and hence Q is noncyclic. But by 8.1, Q is of symplectic type. Thus by 23.9 in [1], $Q = Q_1 * Q_2$, where if p is odd, $Q_1 \cong p^{1+2}$ and Q_2 is cyclic, while if $p = 2$ then $Q_1 \cong Q_8$ and Q_2 is cyclic or dihedral. In particular if p is odd then

$O^{p'}(Out(Q)) \cong SL_2(p)$, while if $p = 2$ then either $O^{2'}(Out(Q)) \cong S_3 \times \mathbf{Z}_{2^n}$ or $Q \cong Q_8 * D_8$ and $O^{2'}(Out(Q)) \cong S_5$. So as $Q = O_p(H)$ and $H = \langle\Lambda(H)\rangle$, $H/Q \cong SL_2(p)$ if p is odd, $H/Q \cong S_3$ if $p = 2$ and Q is not $Q_8 * D_8$, and $H/Q \cong S_3$, D_{10}, A_5, or S_5 if $Q \cong Q_8 * D_8$.

Suppose p is odd. Then $H/Q \cong SL_2(p)$. Hence $H = QC_H(t)$ where t is an involution with $C_H(t) \cong SL_2(p) \times Z(Q)$ and $[t, Q] \cong p^{1+2}$. So as $H = \langle\Lambda(H)\rangle$, $Z(Q) = X$ and $Q = [Q, t]$. Now as $m_2(P) > 2$, $J(P) \cong E_{p^3}$ and if $p > 3$ there is $A \in \mathcal{E}_2^p(P)$ with $A \not\leq Q$ and $A \not\leq J(P)$, so that $A = C_H(A)$, contradicting (ii). Thus if p is odd, (1) holds.

So take $p = 2$. As $m_2(P) \geq 3 > m_2(Q)$, there is an involution $t \in P - Q$, and by Baer-Suzuki, t inverts some R of odd prime order r. If $r = 3$ then t acts on $[R, Q] \cong Q_8$ and $\langle t\rangle R[Q, R] \cong GL_2(3)$. In particular if $H/Q \cong S_3$, then keeping (ii) in mind, we conclude (2) holds.

(8.4): *One of the following holds:*

(1) $X \trianglelefteq G$.

(2) *$F^*(G)$ is the direct product of p components permuted regularly by X and isomorphic to L_X.*

(3) *G is almost simple.*

(4) *$F^*(G) = M \times L_X$ with M a component of G of p-rank 1, $X \leq M$, and 8.2.1 holds, so L_X has a strongly p-embedded subgroup.*

Proof: Assume X is not normal in G. Let $Q = \Omega_1(O_p(G))$. Then

$$Q_0 = \Omega_1(N_Q(X))X \leq \Omega_1(O_p(H)),$$

so by 8.1, $Q_0 = 1$ or X. Thus $Q = 1$ or X and as X is not normal in G, $Q = 1$. Hence $E(G) = F^*(G)$.

Suppose $F^*(\bar{H}) = O_p(\bar{H})$. Then by 8.3, $X = \Omega_1(Z(P))$, so $P \in Syl_p(G)$. Further if $\{L_1, \ldots L_n\}$ are the components of G, then as $X = \Omega_1(Z(P))$, P is transitive on these components and the projection X_i of X on L_i is of order p. But then $X_1 \times \cdots \times X_n \trianglelefteq H$, so by 8.1, $n = 1$ and G is almost simple.

Thus we may assume $F^*(\bar{H}) \neq O_p(\bar{H})$, so that 8.2 applies. By 31.17 in [1], $L_X \leq E(G)$. Let L be a component of G. By 31.18 in [1] one of the following holds:

(a) $L = [L, X]$.

(b) $\langle L^X\rangle$ is the direct product of p components permuted regularly by X and isomorphic to a component of L_X.

(c) $L \trianglelefteq L_X$.

As $E(G) = F^*(G)$ we may choose L so that (a) or (b) holds; in either case let $M = \langle L^X \rangle$. If $M = F^*(G)$ then (2) or (3) holds and we are done, so assume not. Then $F^*(G) = M \times M_1$.

If X induces inner automorphisms on M then the projection X_1 of X on M with respect to the decomposition $M \times C_G(M)$ is contained in $\Omega_1(O_p(H))$, so by 8.1, $X = X_1$ and $M = L$. Then $M_1 \trianglelefteq E(H)$ with $X \not\leq M_1$, so 8.2.1 holds with $M_1 = L_X$. Then as $X = \Omega_1(C_H(L_X))$, $m_p(M) = 1$ and (4) holds.

So X induces outer automorphims on M. In particular $X \not\leq E(G)$ so as $L_X \leq E(G)$, $X \not\leq L_X$ and hence 8.2.1 holds. Also $1 \neq O^{p'}(C_M(X))$, so as 8.2.1 holds, $L_X \leq M$. Similarly $L_X \leq M_1$, contradicting $M \cap M_1 = 1$.

(8.5): *Assume G is almost simple with p odd, and let $M = F^*(G)$ and $L = L_X$. Then one of the following holds:*

1. *$L \cong G(q)$ is of Lie type and Lie rank 1 over $GF(q)$ with q a power of p, $M \cong G(q^p)$, and X induces field automorphisms on M.*
2. *$L \cong A_{2p}$, $X \leq M$, and $M \cong A_{3p}$.*
3. *$p = 3$, $L \cong A_6$, $X \leq M$, and $M \cong J_3$ or $Sp_6(2)$.*
4. *$p = 3$, $L \cong L_2(8)$, $X \leq M$, and either $M \cong Co_3$ or $U_6(2)$ or $O^{3'}(G)$ is $G_2(8)$, ${}^3D_4(2)$, $Sp_4(8)$, $U_3(8)$, or $L_4(8)$, each extended by a field automorphism of order 3.*
5. *$p = 3$, $L \cong A_6$, $M \cong Sp_4(8)$, and X induces field automorphims on M.*
6. *$p = 5$, $L \cong Sz(32)$, and $X \leq M \cong {}^2F_4(32)$.*
7. *$p = 5$, $L \cong {}^2F_4(2)'$, and X induces field automorphisms on $M \cong {}^2F_4(32)$.*
8. *$p = 3$, $C_X \cong GL_2(3)/3^{1+2}$, and $M \cong PSp_4(3)$ or $Sp_6(2)$.*

Proof: Suppose first that $F^*(\bar{H}) = O_p(\bar{H})$. Then by 8.3, $p = 3$ and $\bar{H} \cong GL_2(3)/3^{1+2}$. In particular $P \in Syl_3(G)$ and $m_3(G) = 3$. As $m_3(G) = 3$, 10.6 in [12] says $M \cong A_n$ with $9 \leq n \leq 11$, or $M \cong J_3$, or $M \cong L_3^\epsilon(3)$ or $PSp_4(3)$, or $M \in Chev(q)$ for some prime $q \neq 3$. In the last case we conclude from 14.1 in [12] and the structure of H that $M \in Chev(2)$. Then in any case we conclude from the structure of H that (8) holds.

So assume $F^*(\bar{H}) \neq O_p(\bar{H})$. Then by 8.2, $\bar{C}_X$ is described in 6.2. We observe that if M is sporadic then (3) or (4) holds by 14.4.3 in [12]. (Actually we also need to use the Tables in section 5 of [12] to see that if $p = 3$ and $L_X \cong A_6$ with M sporadic then $O_3(C_X)$ is noncyclic except in case (3).) Thus we assume

M is not sporadic. Hence by 14.4 in [12], $\bar{L}$ is not sporadic.

Suppose that $\bar{L} \cong G(q)$ is of Lie type of Lie rank 1 over $GF(q)$ with q a power of p. Then by 14.16 in [12], either $M \in Chev(p)$ or $p = 3$ and $L \cong A_6$ or $L_2(8)$. In the former case as M is of characteristic p-type, X induces outer automorphisms on M and hence by 1.1, either (1) holds or $p = 3$ and X induces graph or graph-field automorphisms on $M \cong D_4(q)$ or ${}^3D_4(q)$. In the latter case 9.1.2 and 9.1.3 in [12] completes the proof.

Thus we assume $p = 3$, $\bar{L}$ is A_6 or $L_2(8)$, and $M \notin Chev(3)$. Then by 14.16 in [12], $M \in Chev(2) \cup Alt$. Of course if $M \in Alt$ then (2) holds, so take $M \in Chev(2)$. If (1) fails then by 7.2 and 9.1 in [12], X induces inner-diagonal automorphisms on M. Then by 14.4 in [12], M is defined over $GF(q)$, where $q = 2$ unless possibly $q = 8$ and $L \cong L_2(8)$. Now by Burgoyne's Tables in section 34 of [12], (3) or (4) holds.

Next suppose $\bar{L} \cong A_{2p}$ with $p > 3$. Then by 14.15.1 in [12], (2) holds.

Suppose $p = 3$ and $\bar{L} \cong L_3(4)$. Then by 14.19.4 and 9.1.3 in [12], X induces field automorphisms on $M \cong L_3(4^3)$ or graph automorphisms on $M \cong D_4(4)$ or ${}^3D_4(4)$. But then by 9.1 in [12], some element of order 3 in $C_M(X)$ induces an outer automorphism on L, a contradiction.

This leaves $p = 5$ and $\bar{L} \cong Sz(32)$ or ${}^2F_4(2)'$. Then by 14.16 in [12], $M \in Chev(2)$. Then by 14.4 in [12], either (7) holds or the extended Dynkin diagram of the algebraic defining group of M has a B_2 or F_4 subdiagram and M is defined over $GF(q)$ for $q = 32$ or 2, for $\bar{L} \cong Sz(32)$ or ${}^2F_4(2)'$, respectively. By 14.6 in [12], M is not classical, so M is $F_4(q)$ or ${}^2F_4(q)$. In the latter case (6) holds by 14.10.9 in [12]. In the former we check directly that no element of M of order 5 has a component of type L.

(8.6): *Assume G is almost simple and $p = 2$. Let $M = F^*(G)$ and $L = L_X$. Then one of the following holds with q even:*

(1) $L \cong L_2(q)$, $M \cong L_2(q^2)$, *and X induces field automorphisms on M.*
(2) $L \cong L_2(q)$, $L \cong L_3^{\epsilon}(q)$, *and X induces graph automorphims on M.*
(3) $L \cong U_3(q)$, $M \cong L_3(q^2)$, *and X induces graph-field automorphims on M.*
(4) $L \cong Sz(q)$, $M \cong Sp_4(q)$, *and X induces graph automorphims on M.*
(5) $L \cong L_2(8)$ *and X induces graph automorphims on* $M \cong G_2(3)$.
(6) $L \cong L_2(4)$ *and* $XM \cong S_7$
(7) $L \cong L_2(4)$, $X \leq M$, *and* $M \cong J_1$.
(8) $H \cong A_5/Q_8 * D_8$ *and* $G \cong J_2$ *or* J_3.

(9) *$H \cong GL_2(3)YD_{2^8}$ and $M \cong Aut(L_3(3))$.*

Proof: If $F^*(\bar{H}) = O_2(\bar{H})$ then by 8.3, X is 2-central in G and H is described in 8.3. Then we conclude (8) or (9) holds.

So assume $F^*(\bar{H}) \neq O_2(\bar{H})$. Similarly if 8.2.2 or 8.2.3 holds then X is 2-central and we obtain a contradiction from the structure of H. Finally if 8.2.1 holds, we appeal to [13] to conclude either $M \in Chev(2)$ and X induces outer automorphisms on M or one (5)–(7) holds. Then in the former case we appeal to [9].

9. The proof of Theorem 2

In this section we prove Theorem 2. Our original proof of Theorem 2 was longer and less elegant than the one given here. This proof was suggested by Yoav Segev.

Throughout this section we assume the hypotheses of Theorem 2. In addition let $\Lambda = \Lambda_p(G)$, $R = O_{p'}(G)$, and $\bar{G} = G/R$. We may assume $G = \langle \Lambda \rangle$.

(9.1): $m_p(G) > 2$.

Proof: As Λ is connected and $O_p(G) = 1$, 6.1 and 6.2 say $m_p(G) \geq 2$. Then by 6.6, $m_p(G) > 2$.

Let $K = K_p(G)$ and $\mathcal{G}$ the graph on the 1-simplices of K with s adjacent to t if $s \cup t$ is a 2-simplex. Let $\mathcal{C}$ be the set of connected components of $\mathcal{G}$ and for $C \in \mathcal{C}$ let $F(C)$ be the full subcomplex of K on the vertices contained in members of C.

Observe we have a map

$$\psi : \mathcal{G} \to \mathcal{E}_2^p(G)$$
$$\{x, y\} \mapsto xy$$

which induces a bijection $\psi : C \to \psi(C)$ of $\mathcal{C}$ with the set of connected components of $\mathcal{E}_2^p(G)$.

By 2.5 in [19]:

(9.2):

(1) *$F(C)$ is connected for each $C \in \mathcal{C}$.*

(2) *If C, D are distinct members of $\mathcal{C}$ then either $F(C) \cap F(D) = \varnothing$ or $F(C) \cap F(D) = \{x\}$ consists of a single vertex.*

Following Segev in section 2 of [19], let $\Gamma = \Gamma_1 \cup \Gamma_2$ be the bipartite graph with $\Gamma_1 = \{F(C) : C \in \mathcal{C}\}$,

$$\Gamma_2 = \{x \in \Lambda : \{x\} = F(C) \cap F(D),\ C, D \in \mathcal{C}\}$$

and adjacency is equal to inclusion. By Theorem 2.8 in [19]:

(9.3): Γ *is a tree.*

Now G acts as a group of automorphisms of the finite bipartite tree Γ and preserves the bipartition, so G fixes some vertex $\gamma \in \Gamma$. As $O_p(G) = 1$, $\gamma \notin \Gamma_2$ so $\gamma = F(C)$ for some $C \in \mathcal{C}$. Hence G fixes the connected component $\Delta = \psi(C)$ of $\mathcal{E}_2^p(G)$. Further if $\Delta \not\subseteq \mathcal{E}_2^p(G)^\circ$ (*cf.* section 46 of [1]) then $\Delta = \{A\}$ for some $A \in \mathcal{E}_2^p(G)$ and G acts on A contradicting $O_p(G) = 1$. Therefore $\Delta \subseteq \mathcal{E}_2^p(G)^\circ$. Now by 46.7.2 in [1], $\Delta = \mathcal{E}_2^p(\Gamma_{P,2}^\circ(G))^\circ$ and $N_G(\Delta) = \Gamma_{P,2}^\circ(G)$ for some $P \in Syl_p(G)$. So as G acts on Δ we have:

(9.4): $G = \Gamma_{P,2}^\circ(G)$ *and* $\Delta = \mathcal{E}_2^p(G)^\circ$ *is connected.*

Now assume Theorem 2 fails for G. Then:

(9.5): $\mathcal{G}$ *is disconnected.*

Proof: Assume otherwise. Then by 2.3.1 in [19], $\Lambda(x)$ is connected for each $x \in \Lambda$, contrary to our assumption that Theorem 2 fails for G.

Now by 9.4 and 9.5 there is $A \in \mathcal{E}_2^p(G) - \mathcal{E}_2^p(G)^\circ$; that is $\{A\}$ is a connected component of $\mathcal{E}_2^p(G)$, so $\Lambda(A) = F(\psi^{-1}(A))$. Without loss, $A \leq P$. Then $z = \Omega_1(Z(P)) \in \Lambda$ and $\{z\} = F(C) \cap \Lambda(A)$, so $z \in \Lambda_2$.

(9.6): $\{A\} = \mathcal{E}_2^p(C_G(x))$ *for each* $x \in \Lambda(A) - \{z\}$.

Proof: If $g \in C_G(x) - N_G(A)$ then $F(C)zAxA^gF(C)$ is a cycle in Γ, contradicting 9.3.

(9.7): $[z, R] = 1$.

Proof: $[z, R] = \langle [C_R(x), z] : x \in \Lambda(A) - \{z\}\rangle$ and by 9.6, $C_R(x) \leq N_G(A)$, so $[C_R(x), z] = 1$

Notice that as $O_p(G) = 1$, also $O_p(\bar{G}) = 1$ by 9.7. Also by 9.1, $m_p(\bar{G}) > 2$ while by 9.6, $m_p(C_G(A)) = 2$, so $\bar{G}$ satisfies the hypotheses of 7.2. Then 9.6 says that neither case (1) or case (2) of 7.2 holds, so $\bar{G}$ is almost simple.

Further if $p = 2$ then by 7.3, case (b) or (c) of 7.3 holds. But in both cases 9.6 is violated. Therefore p is odd and $\bar{G}$ is described in 7.6. By 9.6, the first two cases of 7.6 do not hold, so $p \geq 5$ and $\bar{L} \cong PSp_4(p)$ or $G_2(p)$. As $G = \langle \Lambda(G) \rangle$ and $Out(\bar{G})$ is of order prime to p, we conclude $\bar{G} = \bar{L}$ and $G = \langle z^G \rangle$, and then by 9.7, $G \leq C_G(R)$ so that G is quasisimple. Now 5.5.3 supplies the final contradiction.

Thus the proof of Theorem 2 is complete.

10. The proof of Theorem 1

In this section G is a finite group and p is a prime.

(10.1): *Let $G = A \times B$ with $\Lambda_p(A) \neq \varnothing \neq \Lambda_p(B)$. Then*

(1) *$K_p(G)$ is connected.*
(2) *$K_p(G)$ is simply connected if and only if $K_p(A)$ or $K_p(B)$ is connected.*

Proof: This follows from 6.7 and 2.1.

(10.2): *Assume $O_p(G) = O_{p'}(G) = 1$. Then one of the following holds:*

(1) *$K_p(F^*(G))$ is simply connected.*
(2) *G is almost simple.*
(3) *$F^*(G) = A \times B$ where A and B are simple with strongly p-embedded subgroups.*

Proof: Assume neither (1) nor (2) holds. As $O_p(G) = O_{p'}(G) = 1$, $F^*(G) = L_1 \times \cdots \times L_r$, where L_i, $1 \leq i \leq r$, are the components of G. Then applying 10.1 to $A_1 \times A_2$ where $A_1 = L_1$ and $A_2 = L_2 \times \cdots \times L_r$, we conclude $K(A_i)$ is disconnected for $i = 1$ and 2. Hence by 6.1, A_i has a strongly p-embedded subgroup, and then by 6.2, A_2 is simple, so that (3) holds.

(10.3): *Let $G = \langle \Lambda_p(G) \rangle$ and $F^*(G) = L = L_1 \times L_2$ the direct product of simple groups L_1 and L_2 with strongly p-embedded subgroups. Then $K_p(G)$ is not simply connected if and only if one of the following holds:*

(1) *$G = L$.*
(2) *$p = 2$ and $G \cong L_1 wr \mathbf{Z}_2$.*
(3) *For $i = 1$ or 2, $p = 3$ and $L_i \cong L_2(8)$ or $p = 5$ and $L_i \cong Sz(32)$. Further if $C_G(L_i) \neq L_{3-i}$ then $L_{3-i} \cong L_i$.*

Proof: Let $K = K(G)$. If $G = G_1 \times G_2$ with $F^*(G_i) = L_i$ then by 10.1, K is simply connected if and only if G_1 or G_2 does not have a strongly p-embedded subgroup. So we may assume G is not a direct product of this form. In particular $G \neq L$.

Suppose L_1 is not normal in G. Then $p = 2$ and some involution $t \in G$ interchanges L_1 and L_2. If $G = \langle t \rangle L$ then $G \cong L_1 wr \mathbf{Z}_2$ and $\Gamma_{P,2}(C_G(t)) \neq C_G(t)$ for $P \in Syl_2(C_G(t))$, so G is not simply connected by 6.3 and Theorem 2. Thus we may assume $G_0 = N_G(L_1) \neq L$. Then $C_{G_0}(t)$ is isomorphic to L_1 extended by an involutory outer automorphism, so in particular by 6.2, $K(C_{G_0}(t))$ is connected. Also $K(G_0)$ is simply connected by induction on the order of G, so by 6.9 applied to $H = G_0$, K is simply connected. Thus we may assume $L_1 \trianglelefteq G$.

As L_1 has a strongly p-embedded subgroup we conclude from 6.2 that $Out(L_1)$ is cyclic. Thus if $L_{3-i} \neq C_G(L_i)$ then we have a decomposition $G = G_1 \times G_2$ as in paragraph one, contrary to the reduction of that paragraph. Thus we have reduced to the case $G = LX$ for some $X \in \Lambda(G)$ inducing outer automorphims on L_1 and L_2. Indeed as X induces outer automorphims on L_i it follows from 6.2 that either $L_i = G_i(q^p)$ is of Lie type and Lie rank 1 with q a power of p and X induces field automorphisms on L_i or $p = 3$ or 5 and $L_i \cong L_2(8)$ or Sz(32), respectively.

Assume $p = 3$ and $L_1 \cong L_2(8)$ or $p = 5$ and $L_1 \cong Sz(32)$. Then for each $X \in \Lambda(G) - \Lambda(L)$, there exists a unique $d(X) \in \Lambda(L_1)$ with $X^\perp \subseteq d(X)^\perp$. Extend d to a map $d : \Lambda(G) \to \Lambda(L)$ by letting $d = id_{\Lambda(L)}$ on $\Lambda(L)$. Then the existence of d and 9.3 in [7] say $K(G)$ and $K(L)$ have the same homotopy type, while by 10.1, $K(L)$ is not simply connected.

Thus we may assume that L_i is not $L_2(8)$, $Sz(32)$ for $p = 3, 5$, respectively.

It remains to show K is simply connected.

Now there exists a group $M = M_1 \times M_2$ with $M_i = L_i X_i \cong L_i X$ and $X = X_1 X_2 \cap G$. Let $D = K(M)$ and $\iota : K \to D$ inclusion. By 10.1, D is simply connected. Thus if we can show ι is locally simply connected in the language of [8], then Theorem 1 in [8] will complete the proof.

But $\iota^{-1}(st_D(s)) = K(C_G(s))$ for each simplex s of D, so in particular as $G \trianglelefteq M$, $\iota^{-1}(st_D(s)) \neq \varnothing$. Further if $S = \langle s \rangle \cap G \neq \varnothing$ then $S \trianglelefteq C_G(s)$, so $K(C_G(s))$ is contractible. Thus it remains to show $K(C_G(Y))$ is simply connected for $Y \in \Lambda(M) - \Lambda(G)$. Let Y_i be the projection of Y on M_i. If $Y_i \neq 1$ for $i = 1$ and 2 then $Z \leq Z(C_G(Y))$ where $Z = Y_1 Y_2 \cap G \in \Lambda(G)$, so again $K(C_G(Y))$ is contractible. Finally if $Y = Y_1$ then $C_G(Y) = C_{L_1}(Y) \times L_2 Z$, where $Z \in \Lambda(G) - \Lambda(L)$ projects on Y_1. In particular $K(L_2 Z)$ is connected, so by 10.1,

$K(C_G(Y))$ is simply connected, completing the proof.

(10.4): *Let* $G = \langle \Lambda_p(G) \rangle$ *and* $F^*(G) = L = L_1 \times L_2$ *where* L_1 *and* L_2 *are simple with strongly p-embedded subgroups. Assume* $\Lambda(x)$ *is connected for each* $x \in \Lambda_p(G)$. *Then* $K_p(G)$ *is not simply connected if and only if either*

(1) $G = G_1 \times G_2$ *with* $F^*(G_i) = L_i$ *and* G_i *having a strongly p-embedded subgroup, or*

(2) $p = 3, 5$, $L_1 \cong L_2(8)$, $Sz(32)$, *respectively, and* $G = LX$ *with* $X \in \Lambda(G)$ *inducing field automorphisms on* L_1 *and* L_2.

Proof: This follows from 10.3, recalling that we saw during the proof of 10.3 that if 10.3.2 holds then $\Lambda(x)$ is disconnected for $x \in \Lambda(G) - \Lambda(L)$.

(10.5): *Let* $O_p(G) = O_{p'}(G) = 1$, $L = F^*(G)$, *and assume* $m_p(L) > 2$ *and* $X \in \Lambda_p(G) - \Lambda_p(L)$ *with* $\Lambda(C_L(X))$ *disconnected. Then one of the following holds:*

(1) $L \cong G(q^p)$ *is of Lie type and Lie rank 1 with* q *a power of* p *and* X *induces field automorphisms on* L.

(2) $p = 2$, $L \cong L_3^\epsilon(q)$ *or* $Sp_4(q)$, q *even, or* $G_2(3)$ *and* X *induces graph automorphisms on* L.

(3) $p = 2$, $L \cong L_3(q^2)$, q *even, and* X *induces graph-field automorphims on* L.

(4) $p > 3$, $L \cong L_p^\epsilon(q)$, *a odd,* $q \equiv \epsilon \mod p$, *and* X *induces diagonal automorphisms on* L.

(5) X *is regular on the* p *components* L_i, $1 \leq i \leq p$, *of* G, *and* L_i *has a strongly* p*-embedded subgroup.*

Further if $\Lambda(X)$ *is connected then one of the following holds:*

(a) $p = 2$ *and* $L \cong L_3(q^2)$, q *even.*

(b) $p > 3$ *and* $L \cong L_p^\epsilon(q^p)$.

(c) G *has* p *components* L_i, $1 \leq i \leq p$, *permuted regularly by* X *and* $L_i \cong G(q^p)$ *is of Lie type and Lie rank 1 with* q *a power of* p.

Proof: By 6.3, XL satisfies the hypotheses of section 7 or section 8. Thus one of (1)–(5) holds by 7.1, 7.2, 7.3, 7.6, 8.4, 8.5, and 8.6. For example most cases in the lemmas are eliminated as $X \not\leq L$ while 8.5.5, 8.5.7, and 8.6.6 do not hold as $m_p(L) > 2$.

So assume $\Lambda(X)$ is connected. Then $\Lambda(C_G(X))$ is not contained in XL, so $Out(L)$ has noncyclic Sylow p-groups in (1)–(4), while in (5) L_i has an outer automorphism a of order p such that $\langle a\rangle L_i$ does not have a strongly p-embedded subgroup. Inspecting the list in (1)–(5), we conclude one of (a)–(c) holds.

We now prove Theorem 1. Assume the hypotheses of Theorem 1 and let $K = K(G)$ and $L = F^*(G)$. By 6.4 and 6.5, K is simply connected if and only if $K(\bar{G})$ is simply connected. Thus we may assume $O_{p'}(G) = 1$. We may also assume $O_p(G) = 1$ as otherwise K is simply connected.

Suppose $K(L)$ is simply connected. By 6.9 we may assume $\Lambda(C_L(X))$ is disconnected for some $X \in \Lambda(G) - \Lambda(L)$. By 6.6, $m_p(L) > 2$. Thus G satisfies the hypotheses of 10.5, so (a), (b), or (c) of 10.5 is satisfied. In cases (a) and (b), $K(L)$ is not simply connected by 5.5 and by Theorem 2, respectively. That is in case (b) there exists $A \in \mathcal{E}_2^p(G)$ with $m_p(C_G(A)) = 2$; namely the preimage of A in $SL_p^\epsilon(q)$ is isomorphic to p^{1+2} and each element of $A^\#$ lifts to an element with p distinct eigenvalues.

So assume (c) holds. Then $J = C_L(X) \cong L_1$ has a strongly embedded subgroup so as $\Lambda(X)$ is connected there exists $Y \in \Lambda(X)$ inducing field automorphisms on J. Hence if $p = 2$ then K is simply connected by 10.3, so we may take p odd. Let G_0 be the subgroup of G fixing each L_i. Then as X is regular on the components of G, $XY \cap G_0 \in \Lambda(G)$ so without loss $Y \leq G_0$. Now we observe that for each $Z \in \Lambda(G_0) - \Lambda(L)$, $\Lambda(C_L(Z))$ is connected while by 10.1, $K(L)$ is simply connected. Thus $K(G_0)$ is simply connected by 6.9. Finally for each $X \in \Lambda(G) - \Lambda(G_0)$, $\Lambda(C_{G_0}(X))$ is connected, so K is simply connected by 6.9.

Thus we may asume $K(L)$ is not simply connected. Then by 10.2 either G is almost simple or $L = L_1 \times L_2$ with L_i containing a strongly p-embedded subgroup for $i = 1$ and 2. In the first case either (1) or (4) holds. In the second (1), (2), or (3) holds by 10.4.

11. A minimal case

In this section p is a prime and G is a finite group such that $G = AH$ where $H = F^*(G)$ is the direct product of simple components L_i, $0 \leq i \leq n$, of order prime to p and permuted transitively by an elementary abelian p-subgroup A of rank at least 3. Let $L = L_1$. Let a_i, $1 \leq i \leq n$, be coset representatives for $B = N_A(L)$ in A with $L^{a_i} = L_i$ and $a_1 = 1$. Write $\alpha_i : L \to L_i$ for the isomorphism $x \mapsto x^{a_i}$. Let $\mathcal{X}$ be a set of B-invariant proper subgroups X of L such that

$$(*) \qquad N_L(X) \cap C_L(B) \leq X$$

Let $\mathcal{F} = \mathcal{F}(\mathcal{X}) = \{\langle X, A\rangle : X \in \mathcal{X}\}$ and consider the geometric complex $\mathcal{C}(G,\mathcal{F})$ over $\mathcal{X}$. (*cf.* Sections 3 and 41 in [1]) Recall $\mathcal{C}(G,\mathcal{F})$ is the simplical complex with vertex set $\bigcup_{F\in\mathcal{F}} G/F$ and simplices $\{U_0,\dots,U_d\}$ such that $\bigcap_{i=0}^{d} U_i \neq \varnothing$. Also G is represented as a group of automorphism on $\mathcal{C}(G,\mathcal{F})$ by right multiplication.

Let $\mathcal{C}(L,\mathcal{X})$ be the geometric simplical complex over $\mathcal{X}$, and let $\mathcal{C}(L,\mathcal{X})^n$ be the geometric product of n copies of $\mathcal{C}(L,\mathcal{X})$. (*cf.* Section 3) Observe H acts as a group of automorphisms of $\mathcal{C}(L,\mathcal{X})^n$ via

$$g = \prod_i g_i\alpha_i : (Xh_1,\dots,Xh_n) \mapsto (Xh_1g_1,\dots,Xh_ng_n).$$

(11.1): *Either*

(1) *A is regular on the components of G, or*

(2) *B is of order p and induces field automorphims on L of Lie type.*

Proof: Assume $B \neq 1$. As $H = F^*(G)$, B is faithful on L. As L has order prime to p but admits an automorphism b of order p, it follows that L is of Lie type, a Sylow p-subgroup of $Out(L)$ is cyclic, and b induces a field automorphism. (*cf.* 1.1) That is (2) holds.

(11.2): *Let $X \in \mathcal{X}$ and $V = \langle X, A\rangle$. Then $N_H(V) = \prod_i X\alpha_i = H\cap V$ and $V = A(H\cap V)$.*

Proof: First $V = A(H\cap V)$ with $H\cap V = \langle X^A\rangle = \prod_i X\alpha_i$. Also $C_H(A) = \prod_i C_L(B)\alpha_i$, and by Hypothesis (*), $C_L(B)\cap N_L(X) \leq X$, so $C_H(A)\cap N_H(V) \leq H\cap V$. Finally by a Frattini argument, $N_H(V) = (H\cap V)(N_H(V)\cap C_H(A)) = H\cap V$.

(11.3): *The map $\phi : (Xg_1,\dots,Xg_n) \mapsto \langle X,A\rangle g$ is an H-equivariant isomorphism $\phi : \mathcal{C}(L,\mathcal{X})^n \to \mathcal{C}(G,\mathcal{F})$ of geometric complexes, where g is the element of H whose projection on L_i is $g_i\alpha_i$.*

Proof: Let $V = \langle X, A\rangle$. By 11.2, $\prod_i X\alpha_i = H\cap V$, so ϕ is a well defined bijection between the set of vertices of $\mathcal{C}(L,\mathcal{X})^n$ and the set of vertices of $\mathcal{C}(G,\mathcal{F})$. Observe also that ϕ is H-equivariant.

Let $s = (U_0,\dots,U_d)$ be a simplex of $\mathcal{C}(L,\mathcal{X})^n$. Then translating by H and using the fact that ϕ is H-equivariant, we may take $U_i = (X_i,\dots,X_i)$. Therefore $\phi(s) = (V_0,\dots V_d)$, where $V_i = \langle X_i, A\rangle$. In particular $\phi(s)$ is a simplex of $\mathcal{C}(G,\mathcal{F})$, so ϕ is a morphism. Similarly ϕ^{-1} is a morphism.

(11.4): *Assume*

(1) $\mathcal{C}(L,\mathcal{X})$ *is a residually connected, simply connected flag complex of dimension at least 2.*

(2) *For each* $\mathcal{Y} \subseteq \mathcal{X}$, $K_p(\langle A, \bigcap_{Y\in\mathcal{Y}} Y\rangle)$ *is simply connected.*

(3) *If* $B \neq 1$ *then* $C_L(B) = \langle C_X(B) : X \in \mathcal{X}\rangle$.

Then $K_p(G)$ *is simply connected.*

Proof: Let $D = \mathcal{C}(L,\mathcal{X})^n$ and $\phi : D \to K(G,\mathcal{F})$ the isomorphism of 11.3. Then $\phi(d) = F(d)h(d)$ for some $F(d) \in \mathcal{F}$ and $h(d) \in H$ and we define $G(d) = F(d)^{h(d)}$ and $\theta(d) = K(G(d))$. Of course these definitions are independent of the choice of coset representative $h(d)$. Notice also that by 11.2, $N_H(F(d)) = H \cap F(d)$, so the map $d \mapsto G(d)$ is injective on objects of any given type but objects of different type may have the same image if distinct members of $\mathcal{X}$ are conjugate in L. Finally observe that as each simplex of $K = K(G)$ is contained in some conjugate of A, $\mathcal{T} = \{\theta(d) : d \in D\}$ is a cover of K.

Let s be a simplex of D, $\theta_s = \bigcap_{d\in s}\theta(d)$, and $G(s) = \bigcap_{d\in s} G(d)$. From the proof of the previous lemma, $G(s)$ is H-conjugate to $\langle X(s), A\rangle$, where $X(s) = \bigcap_{Y\in\mathcal{Y}} Y$ and $\mathcal{Y} \subseteq \mathcal{X}$ is the type of s. Thus as $\theta_s \cong K(G(s))$, θ_s is simply connected by hypothesis (2). Therefore θ is a 1-approximation of K by D in the sense of [8].

Next by hypothesis (1) and 3.3.3, D is simply connected. Thus appealing to Theorem 3 in [8], it suffices to show that if $x \leq A$ is a vertex of K then $\mathcal{T}(x) = \{d \in D : x \in \theta(d)\}$ is connected.

Observe that if $x \neq B$ then $C_H(x)$ is the direct product of the n/p conjugates of $C_{\langle L^a\rangle}(x) \cong L$ under A. On the otherhand if $x = B$ then $C_H(x)$ is the direct product of the n-conjugates of $C_L(x)$ under A.

If $x \in \theta_s$ then $x^G \cap G(s) = x^{G(s)}$. Thus $C_H(x)$ is flag transitive on $\mathcal{T}(x)$. In particular $\mathcal{T}(x)$ is isomorphic to $\mathcal{C}(C_H(x),\mathcal{F}_x)$, where $\mathcal{F}_x = \{C_F(x) : F \in \mathcal{F}\}$. (*cf.* 3.1 in [3]) Now if $x \neq B$ then $\mathcal{C}(C_H(x),\mathcal{F}_x) \cong \mathcal{C}(L,\mathcal{X})^{n/p}$, so that $\mathcal{T}(x)$ is connected in this case. On the otherhand if $x = B$ then $\mathcal{C}(C_H(x),\mathcal{F}_x) \cong \mathcal{C}(C_L(B),\mathcal{X}_B)^n$, where $\mathcal{X}_B = \{C_X(B) : X \in \mathcal{X}\}$, and hence is connected by hypothesis (3) and 3.1, completing the proof.

(11.5): *Assume G and L satisfy the hypotheses of the Conjecture and that the Conjecture holds in all proper sections of G. Assume also that*

(1) $\mathcal{C}(L,\mathcal{X})$ *is a residually connected, simply connected flag complex of dimen-*

sion at least 2.

(2) *If* $B \neq 1$ *then* $C_L(B) = \langle C_X(B) : X \in \mathcal{X} \rangle$.

Then $K_p(G)$ *is simply connected.*

Proof: This follows from 11.4. As proper sections of G satisfy the Conjecture, hypothesis (2) of 11.4 is satisfied via 6.5 applied in an inductive context to any proper subgroup M of G containing A. Notice the hypothesis in 6.5 that $\Lambda(x)$ is connected for $x \in \Lambda = K_p(M)$ holds by 6.1 and 6.2 since $m_p(A) \geq 3$.

12. The proof of Theorem 3

(12.1): *Let* X *be a finite set of order* n *and* Γ *a geometry over* $I = \{1,2,3\}$ *such that there is a bijection* $v_i : X \to \Gamma_i$ *of* X *with* Γ_i *for each* $i = 1,2,3$, *and that* $v_i(x) * v_j(y)$ *for* $i \neq j$ *if and only if* $x \neq y$. *Then*

(1) Γ *has diameter 2 if* $n \geq 3$.

(2) Γ *is simply connected if* $n \geq 5$.

Proof: Part (1) is trivial. Assume $n \geq 5$. Then by (1), Γ has diameter 2, so by 3.3 in [6], it suffices to show squares and pentagons in Γ are trivial. But the objects of distance 2 from $v_1(x)$ are $v_i(x)$, $i = 2,3$ and $v_1(y)$, $y \neq x$. Further $\Gamma(v_1(x), v_1(y)) = \{v_i(z) : z \neq x, y,\ i = 2,3\}$, which is connected as $n \geq 5$. Thus if $p = abcda$ is a square in Γ with $a = v_1(x)$ and $c = v_1(y)$, then p is trivial by 3.4 in [6]. Thus we may take $c = v_2(x)$. But then $b = v_3(z)$ and $c = v_3(w)$, so again p is trivial as we have reduced to a previous case.

Finally if $p = x_0 \cdots x_5$ is a pentagon then we may take $x_0 = v_1(x)$. Then as $d(x_0, x_2) = 2$, $x_2 = v_1(y)$ or $v_i(x)$, $i = 2,3$. But also $d(x_0, x_3) = 2$ so as $x_2 * x_3$ we may take $x_2 = v_1(y)$ and $x_3 = v_2(x)$. But then $v_3(z) \in \Gamma(x_0, x_2, x_3)$ for $z \neq x, y$, so 1.5 in [7] shows p is trivial and completes the proof.

(12.2): *Let* G *be a group 2-transitive on a set* X *of order* $n \geq 5$ *and let* $I = \{1,2,3\}$, x_i, $i = 1,2,3$, *distinct points of* X, $G_i = G_{x_i}$, *and* $\mathcal{F} = (G_i : i \in I)$. *Assume*

(*) $G_i = \langle G_{ij}, G_{ik} \rangle$ *for all distinct* i, j, k *in* I.

Then

(1) $\mathcal{C}(G, \mathcal{F})$ *is a residually connected geometric complex.*

(2) $\Gamma(G, \mathcal{F})$ *satisfies the hypotheses of 12.1, so* $\Gamma(G, \mathcal{F})$ *is simply connected.*

(3) *If* G *is 3-transitive on* X *then* $\mathcal{C}(G, \mathcal{F})$ *is the flag complex of* $\Gamma(G, \mathcal{F})$.

Proof: As G is 2-transitive on X, G_i is maximal in G, so $G = \langle G_i, G_j \rangle$ for all $i \neq j$. This together with hypothesis (*) is equivalent to the residual connectivity of $\mathcal{C} = \mathcal{C}(G, \mathcal{F})$; *cf*. 3.2 in [3].

So (1) is established and visibly $\Gamma = \Gamma(G, \mathcal{F})$ satisfies the hypotheses of 12.1, and hence is simply connected by 12.1. Thus we may assume G is 3-transitive on X. Notice that hypothesis (*) is automatically satisfied in this case since G_i is 2-transitive on $X - \{x_i\}$, so G_{ij} and G_{ik} are maximal in G_i. As G is 3-transitive on X, each triangle in Γ is a 2-simplex of $\mathcal{C}$, so $\mathcal{C} = K(\Gamma)$. Hence (3) holds.

We are now in a position to establish Theorem 3. So assume the hypotheses of Theorem 3. We will apply 11.5 to a suitable family $\mathcal{X}$ of subgroups of L. If L is of Lie type and Lie rank at least 3 let $\mathcal{X}$ be the maximal parabolics containing some fixed Borel subgroup of L. Then $\mathcal{C} = \mathcal{C}(L, \mathcal{X})$ is the building of L and hence is a residually connected, simply connected flag complex; see 5.5 for example. Further if $B = N_A(L) \neq 1$ then by 11.1, B is of order p and induces field automorphisms on L, so we may take B to fix each member of $\mathcal{X}$ and of course $C_L(B) = \langle C_X(B) : X \in \mathcal{X} \rangle$. So 11.5 applies and established part (1) of Theorem 3 when L has Lie rank at least 3.

Next assume L has Lie rank 2. Here we choose $\mathcal{X}$ to be the family $\mathcal{F}$ of section 4. Again $\mathcal{C}$ is a residually connected, simply connected flag complex by 4.1 and 4.2. As above if $B \neq 1$ then we may choose B to fix each member of $\mathcal{X}$ and 11.5.2 is satisfies. Thus 11.5 completes the proof of part (1) of Theorem 3.

In the remaining cases L is 3-transitive on a set X of order $n \geq 5$, so we can appeal to 12.2. As we observed during the proof of 12.2, the 3-transitivity of L on X insures that hypotheses (*) of 12.2 is satisfied. Further if $B \neq 1$ then by 11.2, $L \cong L_2(q^p)$ and B induces field automorphisms on L, so we may choose B to fix each member of $\mathcal{X}$ and hypothesis 11.5.2 is satisfied. By 12.2, hypothesis 11.5.1 is satisfied. Thus 11.5 completes the proof of Theorem 3.

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